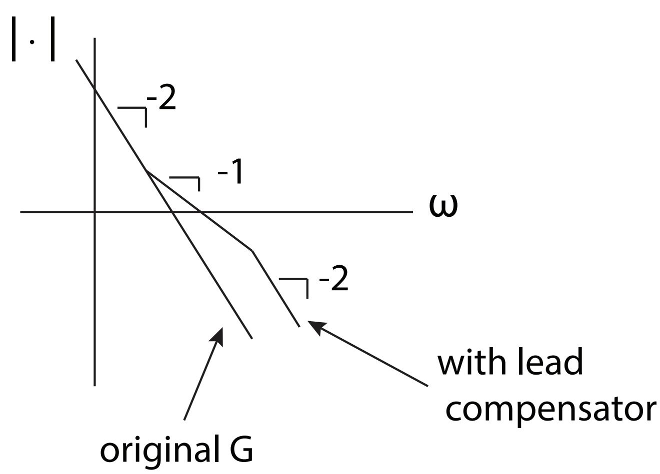
16.06 Principles of Automatic Control
Lecture 25

Lead Compensation

One problem with PD controller is that the gain gets large at high frequencies. So instead use lead compensator

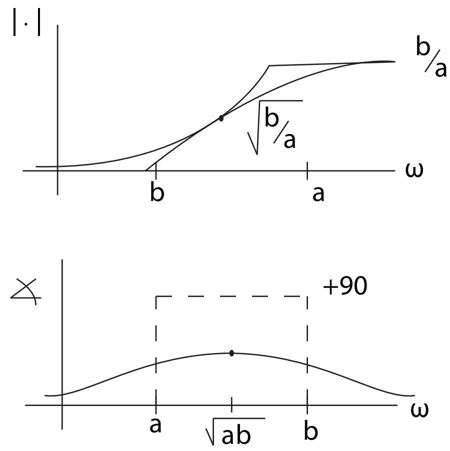
$$K(s) = k\frac{1 + s/a}{1 + s/b}$$

What is the strategy? Look at Bode plot:



To get most phase lead for given lead ratio (b/a), place pole and zero symmetrically around desired crossover. This maximizes average slope near crossover.

The magnitude and phase of a lead compensator are



The phase of the lead compensator is

$$\angle K = \tan^{-1}\omega/a - \tan^{-1}\omega/b$$

The maximum phase lead is

$$\begin{aligned}(\angle K)_{max} &= \tan^{-1}\frac{\sqrt{ab}}{a} - \tan^{-1}\frac{\sqrt{ab}}{b} \\ &= \tan^{-1}\sqrt{b/a} - \tan^{-1}\sqrt{a/b} \\ &= 2\tan^{-1}\sqrt{b/a} - 90^\circ\end{aligned}$$

So to get 60°phase lead (for this example), need lead ratio

$$\frac{b}{a} = 13.92$$

So take

$$\left.\begin{aligned} a &= 0.35\,\mathrm{r/s} \\ b &= 4.9\,\mathrm{r/s}\end{aligned}\right\}\text{ symmetric about } \omega_c = 1.3$$

As $\omega_c = 1.3$ r/s,

$$\begin{aligned}|G| &= \frac{1}{1.3^2} = 0.5917 \\ |K| &= k \cdot \sqrt{\frac{b}{a}} = 3.73k\end{aligned}$$

Therefore, $|GK| = 1 \quad \Rightarrow \quad k = 0.453$. The compensator is then

$$K(s) = 0.453\,\frac{1 + s/0.35}{1 + s/4.9}$$

Again from Matlab,

$$\begin{aligned} t_r =& 0.91\,\text{sec, good!} \\ M_p =& 19\%\text{, not to spec.} \end{aligned}$$

Lead Compensation to achieve minimum K_p.

The example is similar to, but not identical to, FPE example 6.15.

The problem is to control the plant

$$G(s) = \frac{1}{(1 + s/0.5)(1 + s)(1 + s/2)}$$

So that

$$\begin{aligned} K_p =& 9 \\ \text{PM} \geqslant& 30^\circ \end{aligned}$$

$K_p = 9$ guarantees only 10% tracking error in steady state; PM $\geqslant 30^\circ$ensures a minimum stability margin.

First, consider a proportional controller, with unity feedback

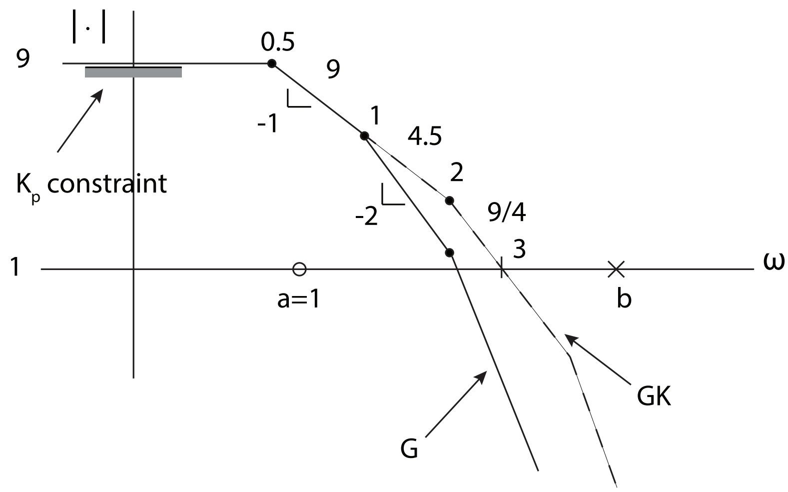
$$K(S) = k_p = 9 \qquad (\text{since } G(0) = 1).$$

Then the control system has

$$\begin{aligned} K_p =& 9 \qquad (\text{as required}) \\ \text{PM} =& 7.1^\circ \qquad (\text{too low}) \end{aligned}$$

So need to add lead compensation to increase PM.

Bode:



To get better slope at crossover, add lead compensator. It's *convenient* to place zero at $s = -1$, since this cancels plant pole, and makes Bode plot simpler. Working out the geometry, this puts the cross-over at

$$\omega_c = 3\,\mathrm{r/s}$$

at least using the straight line approximation.

Our goal is to use the smallest b that meets specs. At crossover, the phase is

$$\begin{aligned}\angle GK &= -\tan^{-1}(\frac{3}{0.5}) - \tan^{-1}(\frac{3}{2}) - \tan^{-1}(\frac{3}{b}) \\ &= -136.9^\circ - \tan^{-1}(\frac{3}{b}) \\ &= -150^\circ \qquad \text{(for PM = 30}^\circ\text{)}\end{aligned}$$

Solving for b,

$$b = 12.8\,\mathrm{r/s}$$

So trial controller is

$$K(s) = 9\,\frac{1+s}{1+s/12}$$

Using Matlab, found

$$\omega_c = 2.62\,\text{r/s}$$
$$\text{PM} = 36.5^\circ$$

Phase margin is larger than required, so can reduce phase lead by 6.5° at (new) ω_c.

$$\begin{aligned}\angle GK &= -\tan^{-1}\left(\frac{2.62}{0.5}\right) - \tan^{-1}\left(\frac{2.62}{2}\right) - \tan^{-1}\left(\frac{2.62}{b}\right) \\ &= -150^\circ\end{aligned}$$

Solving for b, we have

$$b = 8$$

So new controller is

$$K(s) = 9\,\frac{1+s}{1+s/8}$$

which has

$$\omega_c = 2.58\,\text{r/s}$$
$$\text{PM} = 30.9^\circ$$

DONE! The step response is shown in the figure below. Note that M_p is larger than would be expected ($\approx$ 37%) given the PM. This is typical of systems with modest k_p.