16.06 Principles of Automatic Control
Lecture 23

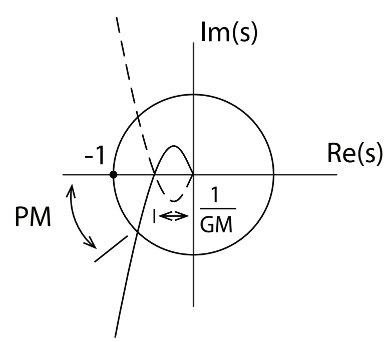
Stability Margins

Stability margins measure how close a closed-loop system is to instability, that is, how large or small a change in the system is required to make it become unstable. The two commonly used measures of stability are the *gain margin* and the *phase margin*.

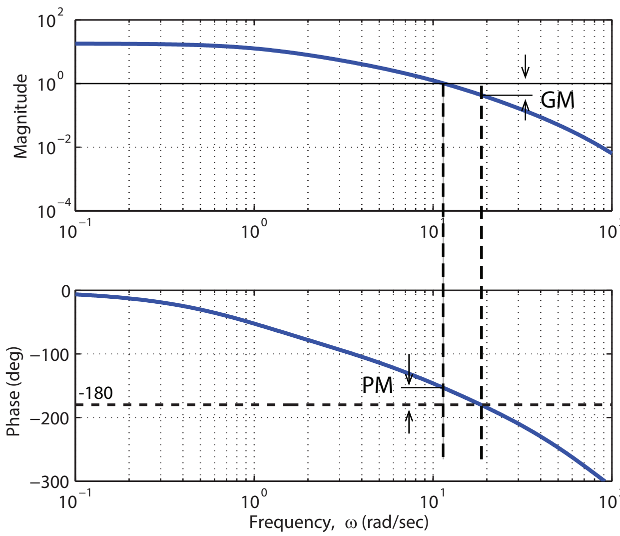
- The gain margin (GM) is the factor by which the gain can be increased before the system becomes unstable.
- The phase margin (PM) is the amount of additional phase lag that would make the phase be -180° where $|KG(j\omega)| = 1$.

The GM and PM are important not only because they measure how close the closed-loop system is to instability, but also because they (but especially the PM) can be used to predict the transient behavior of the closed-loop system.

Gain and phase margin on Nyquist diagram:



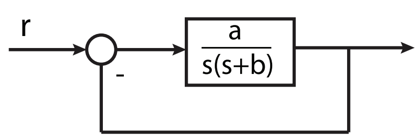
GM and PM on Bode diagram:



Relationship between PM and damping

When the phase margin is small, the closed-loop system is close to instability, so that there will be closed-loop poles near the $j\omega-$axis. That is, low PM $\Rightarrow$ low damping ratio.

This result can be made explicit by considering the closed-loop system



The closed-loop transfer function is

$$T(s) = \frac{a}{s^2 + bs + a}$$

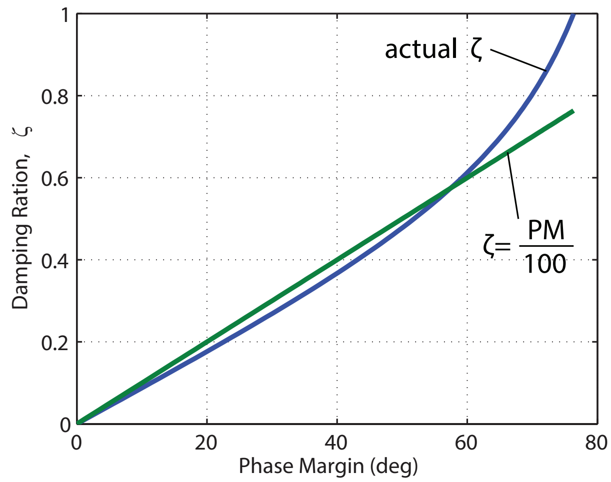
So,

$$\omega_n = a$$
$$\zeta = \frac{b}{2\sqrt{a}}$$

Can show that, *for this system*,

$$\text{PM} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^2}-2\zeta^2}}\right)$$

The functional form isn't really important - the important point is that ζ is nearly a linear function of PM:



So can often predict (effective) damping ratio using approximation

$$\zeta \approx \frac{\text{PM}}{100} \quad \text{(PM in degrees)}$$

Even when system is not second order, PM is a good predictor of peak overshoot (M_p), and resonant peak magnitude (M_r). PM is often specified as a design requirement.

Bode's Gain-Phase Relationship

We saw that for poles and zeros in the left-half-plane, the phase of $G(j\omega)$ is proportional to the slope of the magnitude curve (on a log-log scale), but smeared-out. That is,

$$\angle G(j\omega) \approx 90^\circ \times \text{ slope of } |G|$$

This idea can be made precise via Bode's gain-phase theorem:

> *For any stable, minimum phase system, the phase of $G(j\omega)$ can be determined uniquely from the magnitude of $G(j\omega)$.*

The phase is in fact given by

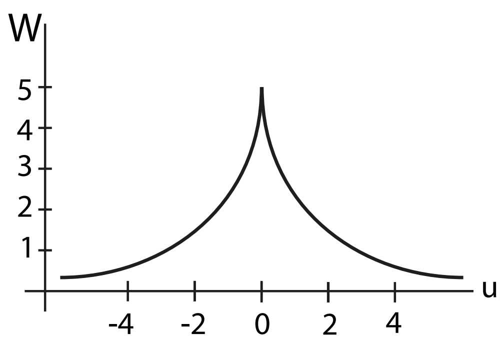
$$\angle G(j\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dM}{du} W(u) du$$

where

$$\begin{aligned}
M &= \log|G(j\omega)| \quad \text{(natural log)} \\
u &= \log(\omega/\omega_0) \\
\frac{dM}{du} &= \text{slope of Bode plot magnitude} \\
W(u) &= \text{weighting function} \\
&= \log(\coth(\frac{101}{2}))
\end{aligned}$$

Note that this is a funny sort of convolution - we are convolving a weighting function with the slope of another function, but working on logarythmic axes!

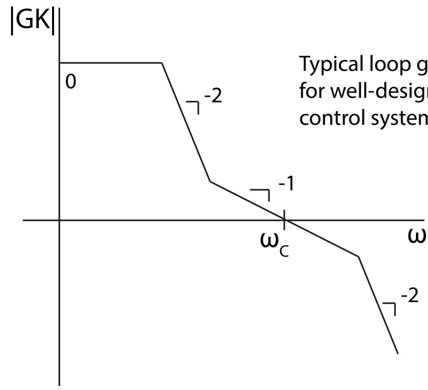
The weighting function looks like:



Note that 92% of area of $W(u)$ is within ± 1 decade of the center. So the phase is nearly completely determined by the slope of M within ~1 decade.

Why is this result important? It implies that in almost every case, a well-designed control loop will have a magnitude plot with slope -1 at the crossovr frequency![1]

[1] Actually, in some cases, the slope might be +1, but this is rare.



In this case, the phase at cross-over will be a weighted average of -90° (weighted a lot), -180° (weighted some), and 0° (weighted hardly at all). So the phase will be between -90° and -180°, with probably reasonable PM.

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