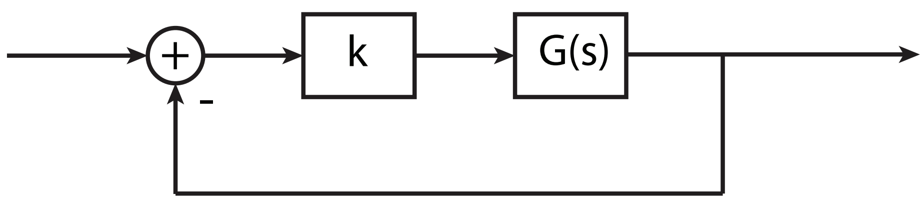
16.06 Principles of Automatic Control
Lecture 20

Bode Plots With Complex Poles

Suppose we have a proportional feedback system:



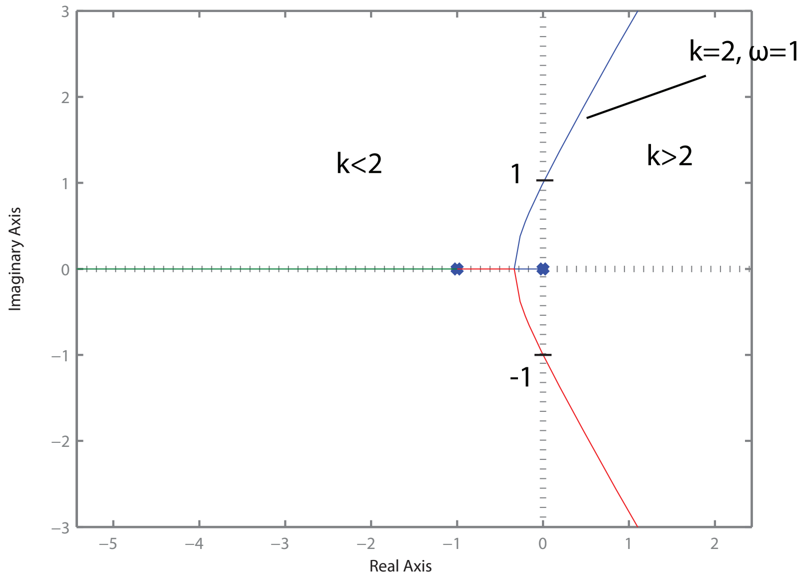
What values of k will lead to instability? Before we answer that, let's find out what values lead to *neutral stability*. Take, as an example,

$$G(s) = \frac{1}{s(s+1)^2}$$

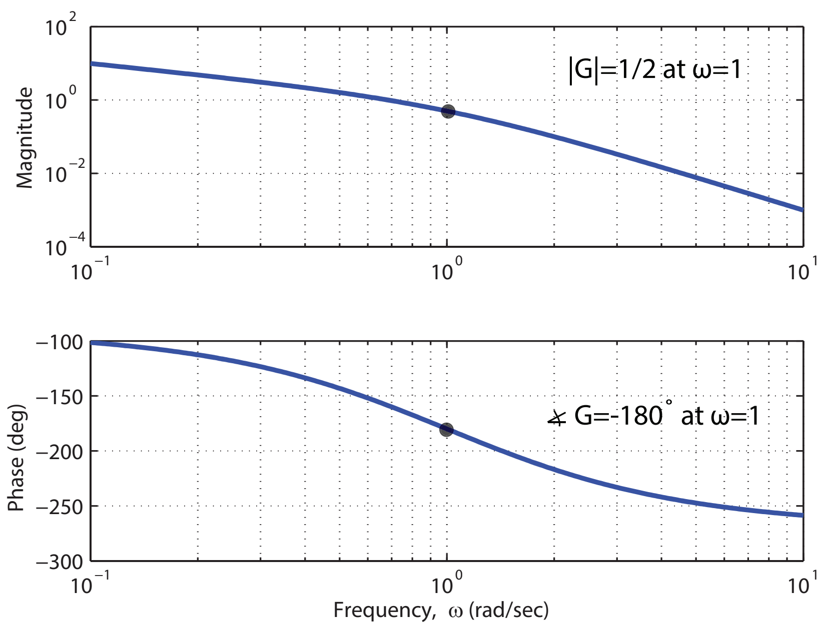
Using root locus and Routh, we can deduce that the C.L. system is stable for

$$0 < k < 2$$

The root locus diagram is:



So neutral stability occurs for $k = 2$, corresponding to closed-loop poles at $\omega = \pm 1$. This result may be seen clearly on the Bode plot for this system.



Recall that the root locus condition is that

$$kG = 1$$

or

$$G = -1/k$$

For there to be a closed loop pole on the $j\omega$ axis for $k > 0$, we must have that two conditions hold. First, G must have phase of $-180°$. The only frequency at which this happens is $\omega = 1$ rad/sec. Second, we must have that

$$|kG| = |-1| = 1$$
$$\Rightarrow k = \frac{1}{|G|}$$

In this case, $|G| = 1/2$ at $\omega = 1$, so $k = 2$ is the required gain to place a pair of poles on the $j\omega$ axis.

So the Bode plot plays a key role in stability analysis. We already have a partial result:

> If the open-loop system $KG(s)$ is stable, and $|KG(j\omega)| < 1$ for all ω such that $\angle KG(j\omega) = 180° (\mod 360°)$, then the closed-loop system is stable.

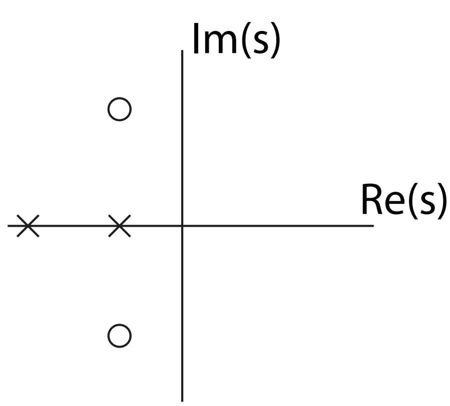
This result follows from our R.L. analysis.

Note that the converse statement is *not* true,that is, there may be frequencies ω such that $|KG(j\omega)| > 1$ and $\angle KG(j\omega) = 180°$, and yet the closed loop system is stable.

The *Nyquist Criterion* is the Frequency Response analogue of the Routh Criterion - it allows us to count the number of closed-loop, unstable poles. The Nyquist Criterion depends on Cauchy's Principle of the Argument, or simply the argument principle.

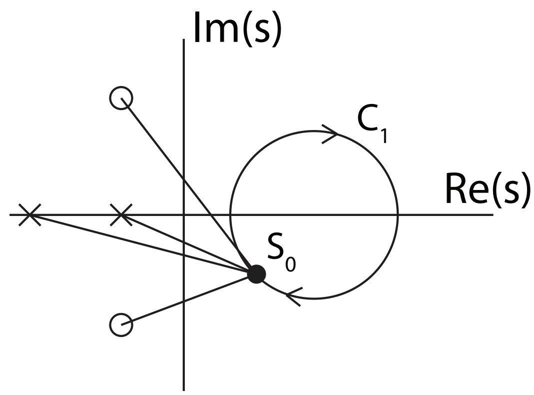
The Argument Principle

Consider a transfer function $H_1(s)$ with pole/zero diagram

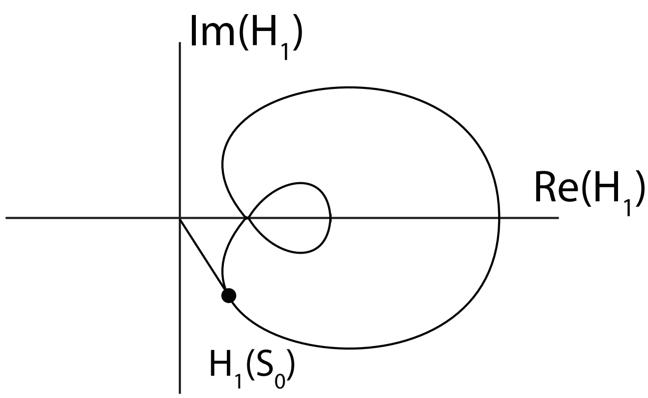


$$H_1(s) = \frac{k\pi(s - z_i)}{\pi(s - p_i)}$$

We are going to evaluate $H_1(s)$ point-by-point around the contour C_1 :



At each point on the contour, we calculate $H_1(s)$ and plot:

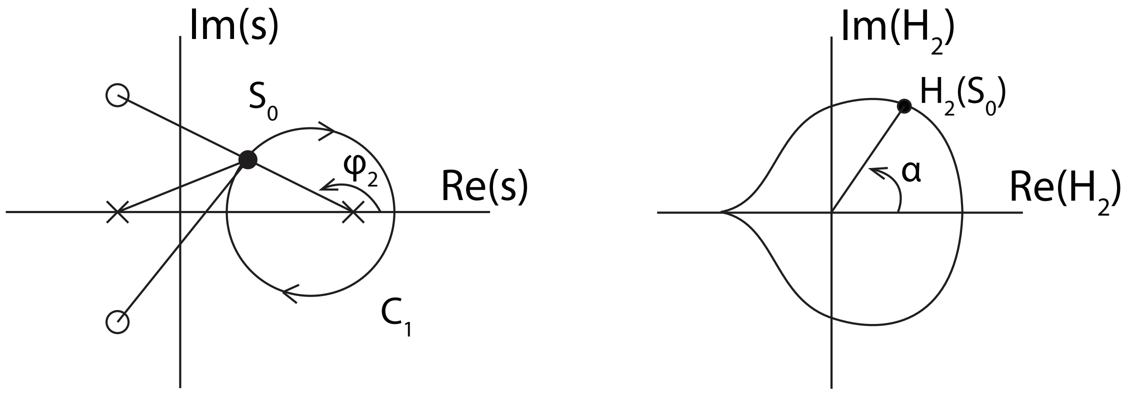


At any point, say s_0, the phase of $H_1(s_0)$ is

$$\begin{aligned}\alpha = \angle H_1(s_0) &= \sum \angle(s_0 - z_i) - \sum \angle(s_0 - p_i) \\ &= \sum \Psi_i - \sum \phi_i\end{aligned}$$

As we go around the contour (in this example), each Ψ_i and ϕ_i increases and decreases, but returns to its original value after completing exactly one circuit.

Consider a second example, H_2 :



In this case, as we move once around C_1, Ψ_i, Ψ_2, and ϕ_1 return to their original values, but ϕ_2 decreases by a net 360°. As a result, $\alpha = \angle H_2$ increases by a net 360°. But this is equivalent to saying that $H_2(C_1)$ encircles the origin exactly once in a clockwise direction.

More generally, the contour map $H_2(C_1)$ encircles the origin counter-clockwise for each pole inside C_1, and clockwise for each zero. More succinctly, for a clockwise contour C_1,

$$\# \text{ of clockwise encirclements of the origin by } H(C_1) = \text{Z - P}$$

where Z = # of zeros of $H(s)$ inside C_1;

and P = # of poles of $H(s)$ inside C_1.