16.06 Principles of Automatic Control
Lecture 13

Root Locus Rules (cont-d)

- **Rule 5** The locus crosses the $j\omega$ axis at points where the Routh criterion shows a transition in the number of unstable roots.

Example:

$$L(s) = \frac{1}{(s+1)^3}$$

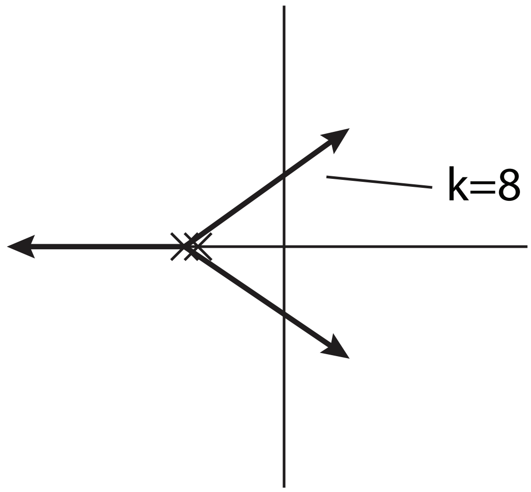
The characteristic equation is

$$s^3 + 3s^2 + 3s + 1 + k = 0$$

The Routh array is

$$\begin{array}{cc} 1 & 3 \\ 3 & 1+k \\ \frac{8-k}{3} & 0 \\ 1+k & 0 \end{array}$$

So the transitions occur at $k = 8, -1$.Look at locus:



For $k = 8$,the characteristic equation is

$$s^3 + 3s^2 + 3s + 9 = 0$$

which has roots at

$$s = -3, \underbrace{\pm\sqrt{3}j}_{\substack{\downarrow \\ j\omega \text{ axis crossing}}}$$

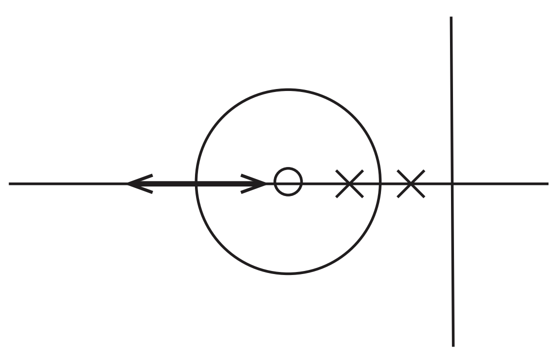
- **Rule 6** The locus will have multiple roots at points on the locus where

$$n(s)\frac{dd(s)}{ds} - d(s)\frac{dn(s)}{ds} = 0$$

(see FPE for details)

Example:

$$L(s) = \frac{s+3}{(s+1)(s+2)}$$



where does locus depart/arrive real axis?

$$\begin{aligned} n(s) =& s+3 \\ d(s) =& s^2+3s+2 \end{aligned}$$

$$\begin{aligned} n'(s) =& 1 \\ d'(s) =& 2s+3 \end{aligned}$$

$$n(s)d'(s) - d(s)n'(s)$$

$$\begin{aligned} =&(s+3)(2s+3) - (s^2+3s+2) \\ =&2s^2+9s+9-(s^2+3s+2) \\ =&s^2+6s+7=0 \\ \Rightarrow s =& -\frac{6}{2} \pm \frac{\sqrt{36-28}}{2} \\ s =& -3 \pm \sqrt{2} \end{aligned}$$

as in recitation!

Lead Compensator Example

For the plant

$$G(s) = \frac{2}{(s+1)(s+2)}$$

find a unity feedback controller with compensator $K(s)$ such that

$$\begin{aligned} t_r \lesssim& 0.5 \text{ sec} \\ M_p \lesssim& 10\% \end{aligned}$$

If the closed loop system is second order, the poles would need to have

$$M_p = 0.10 = e^{-\pi \tan\theta} \Rightarrow \theta = 0.6325 \quad \Rightarrow \zeta = \sin\theta = 0.5912$$

So set $\boxed{\zeta = 0.707}$ to allow some margin.

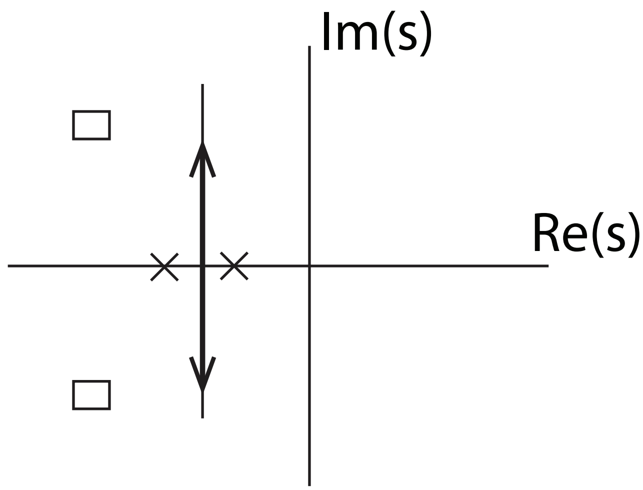
$t_r = \frac{1.8}{\omega_n} \Rightarrow \omega_n \approx 3.6$ rad/sec
This would place poles at

$$s = -2.6 \pm 2.6j$$

To simplify, want poles at

$$s = -3 \pm 3j$$

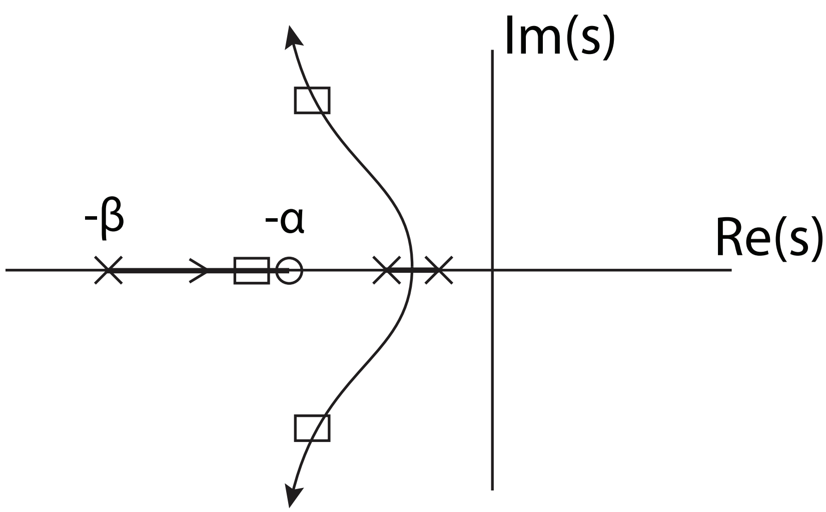
Look at locus with gain only:



So gain only doesn't work - must add *lead compensation:*

$$K(s) = k\frac{(s+\alpha)}{(s+\beta)}$$

where $\alpha < \beta$.
Then rough locus will be

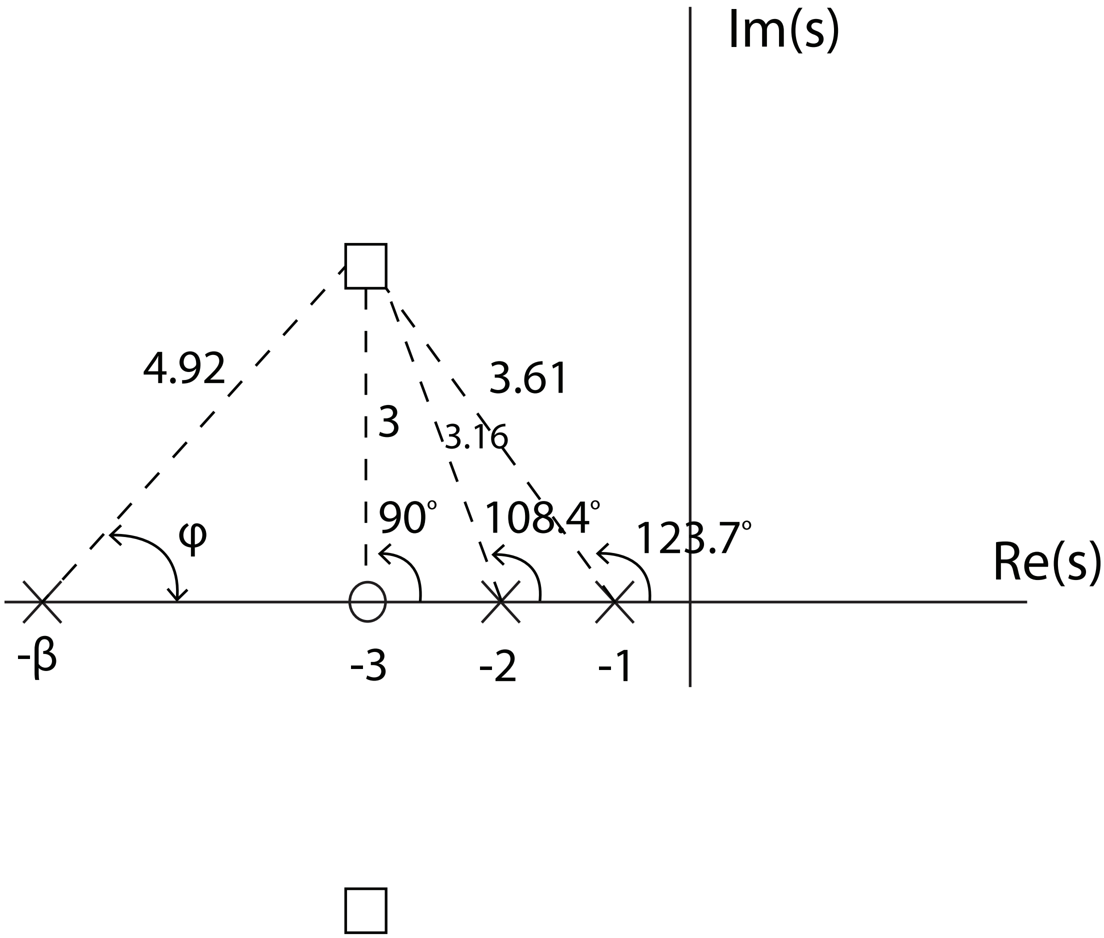


Must choose $\alpha, \beta,\ k$ to make this work.

We have multiple degrees of freedom, so answer is not unique. Let's fix

$$\alpha = 3$$

to guarantee the real closed-loop pole settles faster than complex poles. Then β must be selected to achieve desired angle condition.



16.06 Principles of Automatic Control
Fall 2012