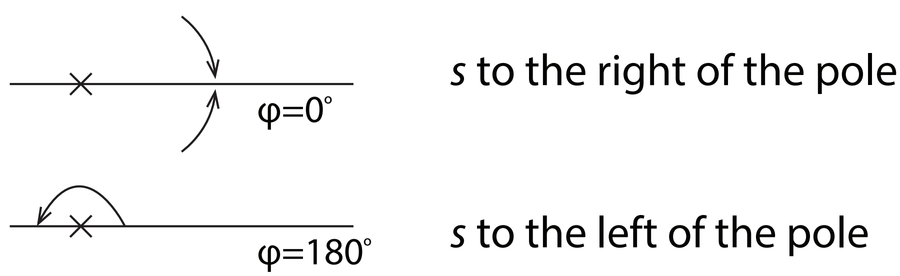
16.06 Principles of Automatic Control
Lecture 12

Root Locus Rules

- **Rule 1** The n branches of the locus start at the n poles of L(s). m branches end at the zeros of L(s). $n - m$ branches end at $s = \infty$.
- **Rule 2** The locus covers the real axis to the left of an odd number of poles and zeros.

To the left of the pole, $\phi = 180°$
To the left of a zero, $\Psi = 180°$.



To the right of a pole, $\phi = 0°$
To the right of a zero, $\Psi = 0°$.
So,

$$\begin{aligned}\angle L(s) &= \sum_{i=1}^{m} \Psi_i - \sum_{i=1}^{n} \phi_i \\ &= m_1 180° - n_1 180° \\ &= (m_1 + n_1)180° - n_1 360° \\ &= 180 + l\,360°\end{aligned}$$

if $m_1 + n_1$ is off, where

$m_1 =$ number of zeros to the right of s

$n_1 =$ number of poles to the right of s

- **Rule 3** For large k, $n-m$ of the loci are asymptotic to the lines emanating from the point $s=\infty$, with angles

$$\theta_l = \frac{180^\circ + 360^\circ \cdot (l-1)}{n-m}, \quad l = 1, ...n-m$$

where $\alpha = \frac{\sum p_i - \sum z_i}{n-m}$.
Why? If $s \to \infty$, $k \to \infty$,then to highest order the equation

$$d(s) + kn(s) = 0$$

becomes

$$s^n + ... + k(s^m + ...) = 0$$

So the solution satisfies

$$s^n \sim -ks^m, \quad (k, s \to \infty)$$

$\uparrow$ "asymptotic to"

$$\Rightarrow s^{n-m} \sim -k$$
$$\Rightarrow s \sim (-k)^{\frac{1}{n-m}}$$
$$= k^{\frac{1}{n-m}} \angle \frac{180^\circ + 360^\circ \cdot (l-1)}{n-m}$$

To get the point $s = \alpha$, do asymptotic analysis with next terms:
Result is that center of pattern is at:

$$s = \frac{\sum p_i - \sum z_i}{n-m}$$

A related rule, not in FPE, is:

- **Rule 3a** If $n-m \geqslant 2$, the centroid of the closed-loop poles is constant, and is at

$$\frac{\sum p_i}{n}$$

To show this, consider a polynomial with roots $z_1, z_2, ...$ The polynomial is then

$$\begin{aligned} &(s-p_1)(s-p_2)...(s-p_n) \\ =&s^n - (p_1 + p_2 + ...p_n)s^{n-1} + ... \end{aligned}$$

Therefore, $a_1 = -\sum p_i$.
Now, the closed loop polynomial is given by

$$\begin{aligned} &d(s) + kn(s) \\ =&s^n + a_1 s^{n-1} + ... + k(s^m + b_1 s^{m-1} + ...) \\ =&s^n + a_1 s^{n-1} + ... + (a_{n-m} + k)s^m + ... \end{aligned}$$

That is, the first term to change in the polynomial is the a_{n-m}term. If $n-m \geqslant 2$, the $a_1$term is unchanged, and the centroid is a constant.
Note that if m poles go to the m zeros z_i, the centroid of the remaining $n-m$ poles must go to

$$\frac{\sum p_i - \sum z_i}{n-m}$$

in agreement with rule 3.

- **Rule 4** The angle(s) of departure of a branch of the locus from a pole of multiplicity q is

$$q\phi_{\text{dep}} = \sum \Psi_i - \sum_{*} \phi_i - 180^\circ - 360^\circ(l-1)$$

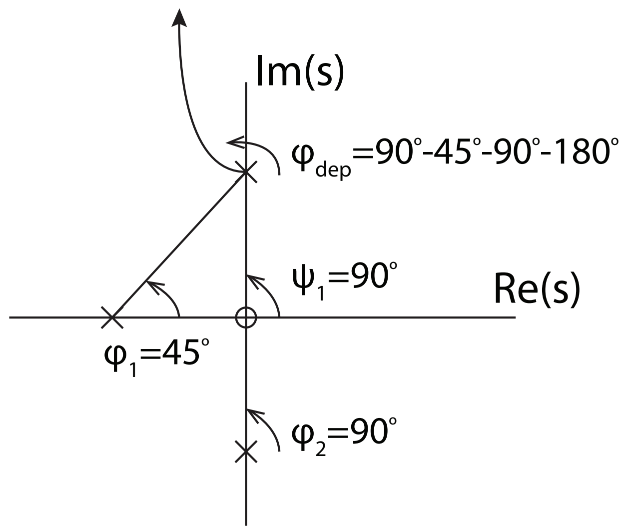
and the angle(s) of arrival of a branch at a zero of multiplicity q is given by

$$q\Psi_{\text{arr}} = \sum \phi_i - \sum_{*} \Psi_i + 180^\circ + 360^\circ(l-1)$$

where the sum $\sum_{*}$excludes to poles (or zeros) at the point of interest.

Example:

$$L(s) = \frac{s}{(s+1)(s^2+1)}$$



$\phi_{\text{dep}} = 90^\circ - 45^\circ - 90^\circ - 180^\circ$
$\phi_{\text{dep}} = -225^\circ = 135^\circ (\text{mod } 360^\circ)$