16.06 Principles of Automatic Control
Lecture 11

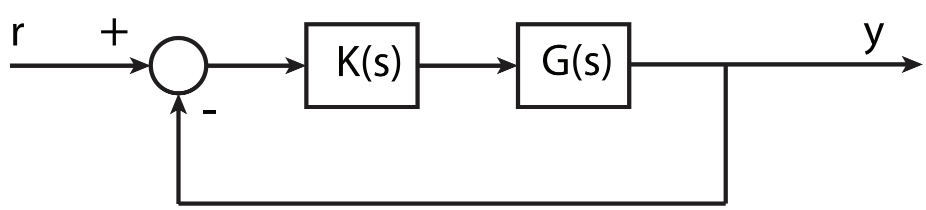
The Root Locus Method

Often, it is useful to find how the closed-loop poles of a system change as a single parameter is varied. To do this, we use the *root locus method*.

Root - root of s polynomial equation

Locus - Set of points (plural - loci)

Consider a typical feedback loop



If both $K(s)$ and $G(s)$ are rational, then the loop gain may be expressed as

$$K(s)G(s) = kL(s)$$

where

$$\begin{aligned}
L(s) &= \frac{n(s)}{d(s)} \\
n(s) &= s^m + b_1 s^{m-1} + ... + b_m \\
&= (s - z_1)(s - z_2)...(s - z_m) \\
&= \prod_{i=1}^{m} (s - z_i) \\
d(s) &= s^n + a_1 s^{n-1} + ... + a_n \\
&= \prod_{i=1}^{n} (s - p_i)
\end{aligned}$$

Then the roots of the closed-loop system occur at:

$$1 + K(s)G(s) = 0 \quad (\star)$$

or

$$1 + kL(s) = 0 \quad (\star)$$

or

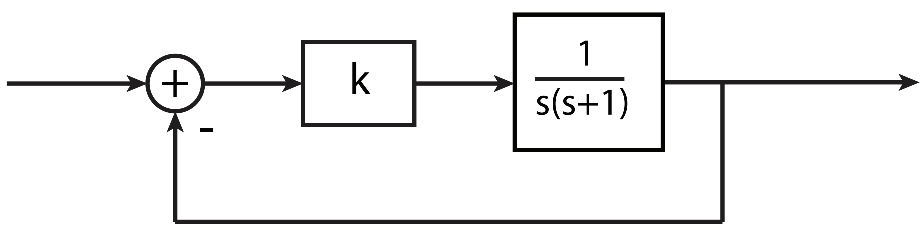
$$L(s) = -\frac{1}{k} \quad (\star)$$

or

$$d(s) + kn(s) = 0 \quad (\star)$$

The root locus is the set of values s for which $(\star)$ holds, and k is any positive real value. (For reasons that will become clear later, this is the definition of the *positive* or *180 degree* locus. Will later define the *negative*, or *0 degree* locus.)

Example:



In this case,

$$\begin{aligned} L(s) =& \frac{1}{s(s+1)}, \quad n(s) = 1 \\ d(s) =& s(s+1) = s^2 + s \\ &\text{zeros:} \quad \text{none} \\ &\text{poles:} \quad p_i = 0, -1. \end{aligned}$$

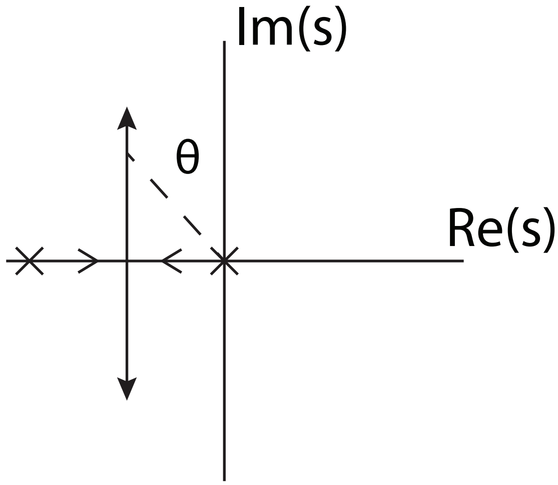
The characteristic equation is:

$$s^2 + s + k = 0$$

Because characteristic equation is quadratic, we can find the roots using the quadratic formula:

$$s = -\frac{1}{2} \pm \frac{\sqrt{1-4k}}{2}$$

When $0 \leqslant k \leqslant \frac{1}{4}$, the roots are real, and between -1 and 0. For $k > \frac{1}{4}$, the roots are complex, with real part $-\frac{1}{2}$, and imaginary part that increases (asymptotically) as $\sqrt{k}$.

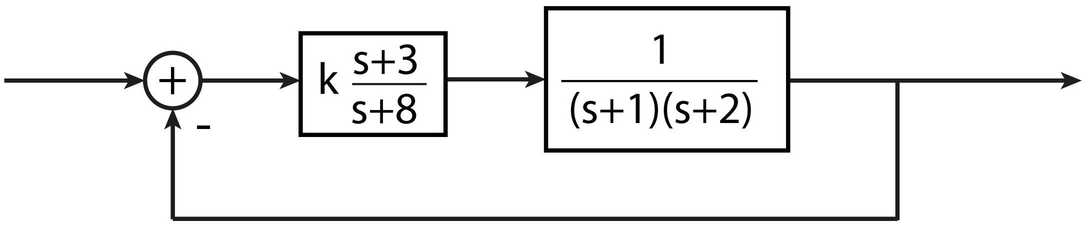


Suppose our goal is to choose k so that $\zeta = \sin\theta = 0.5 \Rightarrow \theta = 30°$.
Looking at the geometry in the figure, the imaginary part is

$$\begin{aligned}
\text{Im}(s) =& \frac{-\text{Re}(s)}{\tan\theta} \\
R(s) =& -\frac{1}{2} \\
\tan\theta =& \frac{\sin 30°}{\cos 30°} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} \\
\Rightarrow \text{Im}(s) =& \sqrt{3}/2 \\
\text{But Im}(s) =& \frac{\sqrt{4k-1}}{2} \\
\therefore \mathbf{k} =& \mathbf{1}
\end{aligned}$$

Example:

What is the root locus of



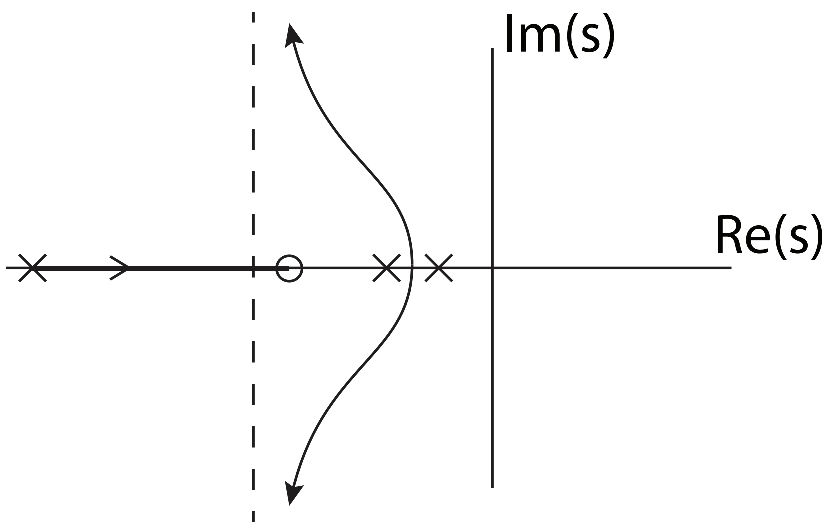
In this problem,

$$L(s) = \frac{s+3}{(s+8)(s+1)(s+2)} = \frac{n(s)}{d(s)}$$

The characteristic equation is

$$(s+8)(s+1)(s+2) + k(s+3) = 0$$

Because the polynomial is cubic, we can't find the roots (easily) in closed form. Nevertheless, can sketch the root loci using root loci sketching rules:



With a little practice, you should be able to sketch root loci very rapidly.

Guidelines for Sketching Root Locus

Will give rules for $k > 0$.
For $k > 0$ and $1 + kL(s) = 0$ must have that

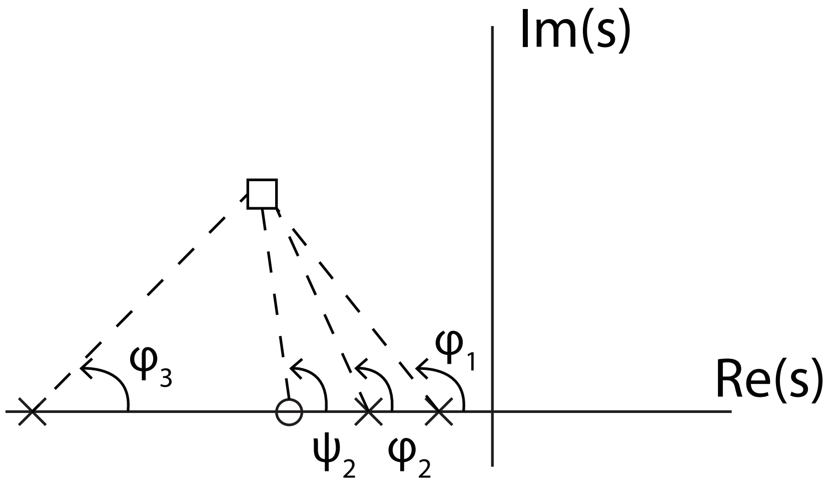
$$L(s) = -\frac{1}{k} = \text{negative real number}$$

That is, the phase of the L(s) must be:

$$\angle L(s) = 180^\circ + l \cdot 360^\circ, \text{ where l is an integer.}$$

This is the root locus *phase condition*, and the reason we call the locus for $k > 0$ the 180°locus.

Consider the example above:



The phase of L(s) is given by

$$\angle L(s) = \Psi_1 - \phi_1 - \phi_2 - \phi_3$$

To see if a given point is on the locus, could measure all the angles, add/subtract, and test result. This used to be done mechanically with a "spirule". However, it's only important to be able to sketch general shapes; Matlab can do the rest.

Root Locus Rules

Rule 1

The n branches of the locus start at the n points of L(s). m branches end at the zeros of L(s). $n - m$ branches end at $s = \infty$.

Rule 2

The loci cover the real axis to the left of an odd number of poles and zeros.
To the left of the pole, $\phi = 180^\circ$
To the left of a zero, $\Psi = 180^\circ$.