Area and Bending Inertia of Airfoil Sections

Calculation of the vertical deflection of a wing requires knowing the spanwise bending stiffness distribution $EI(y)$ along the primary axis of loading. For a wing made of a uniform solid material, the modulus E is a simple scaling factor. The moment of inertia of the airfoil cross-sections about the bending axis x (called the *bending inertia*), is then related only to the airfoil shape given by the upper and lower surfaces $Z_u(x)$ and $Z_\ell(x)$. As shown in Figure 1, both the area A and the total bending inertia I are the integrated contributions of all the infinitesimal rectangular sections, each dx wide and $Z_u - Z_\ell$ tall. The inertia of each such section is appropriately taken about the *neutral surface* position $\bar{z}$ defined for the entire cross section.

$$A = \int_0^c \left[Z_u - Z_\ell \right] dx \tag{1}$$

$$\bar{z} = \frac{1}{A} \int_0^c \frac{1}{2} \left[Z_u^2 - Z_\ell^2 \right] dx \tag{2}$$

$$I = \int_0^c \frac{1}{3} \left[(Z_u - \bar{z})^3 - (Z_\ell - \bar{z})^3 \right] dx \tag{3}$$

These relations assume that the bending deflection will occur in the z direction, which is a good assumption if the x axis is parallel to the airfoil's chord line.

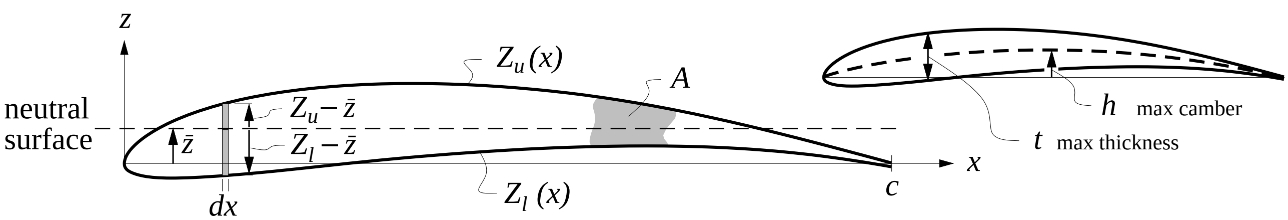

Figure 1: Quantities for determining and estimating the bending inertia of an airfoil section.

Although equations (1) – (3) can be numerically evaluated for any given airfoil (e.g. using XFOIL's `BEND` command), this is unnecessarily cumbersome for preliminary design work, where both A and I are needed for possibly a very large number of candidate airfoils or wings.

For the purpose of approximating A and I, we first define the maximum thickness t, and maximum camber h, in terms of the upper and lower surface shapes. We also define the corresponding thickness and camber ratios τ and ε.

$$t = \max \{ Z_u(x) - Z_\ell(x) \} \tag{4}$$

$$h = \max \{ [Z_u(x) + Z_\ell(x)] /2 \} \tag{5}$$

$$\tau \equiv t/c$$

$$\varepsilon \equiv h/c$$

Examination of equation (1) indicates that A is proportional to $t\,c$, and examination of (3) indicates that I is proportional to $c\,t(t^2 + h^2)$. This suggests estimating A and I with the following approximations.

$$A \simeq K_A\, c\, t = K_A\, c^2\, \tau \tag{6}$$

$$I \simeq K_I\, c\, t\, (t^2 + h^2) = K_I\, c^4\, \tau(\tau^2 + \varepsilon^2) \tag{7}$$

The proportionality coefficient can be evaluated by equating the exact and approximate A and I expressions above, e.g.

$$K_A \quad \leftarrow \quad \frac{1}{c^2\,\tau} \int_0^c \Big[Z_u - Z_\ell \Big] \; dx \tag{8}$$

$$K_I \quad \leftarrow \quad \frac{1}{c^4\,\tau(\tau^2+\varepsilon^2)} \int_0^c \frac{1}{3} \Big[(Z_u - \bar{z})^3 - (Z_\ell - \bar{z})^3 \Big] \; dx \tag{9}$$

Evaluating these expressions produces nearly the same K_A and K_I values for most common airfoils:

$$K_A \;\simeq\; 0.60 \tag{10}$$

$$K_I \;\simeq\; 0.036 \tag{11}$$

Therefore, the very simple approximate equations (6) and (7), with K_A and K_I assumed fixed, are surprisingly accurate. Hence, they are clearly preferred for preliminary design work over the exact but cumbersome equations (1), (2), (3).