

# Wing Bending Calculations

## Lab 10 Lecture Notes

### Nomenclature

$y$	spanwise coordinate	$\eta$	normalized spanwise coordinate ( $= 2y/b$ )
$q$	net beam loading	$c$	local wing chord
$\mathcal{S}$	shear	$S_{\text{wing}}$	wing area
$M$	bending moment	$b$	wing span
$\theta$	deflection angle ( $= dw/dx$ )	$\lambda$	taper ratio
$w$	deflection	$E$	Young's modulus
$\kappa$	local beam curvature	$\delta$	tip deflection
$L'$	lift/span distribution	$N$	load factor
$m'$	wing mass/span distribution	$L$	lift
$I$	bending inertia	$W$	weight
$i$	spanwise station index	$g$	gravitational acceleration
$n$	last station index at tip	$(\ )_o$	quantity at wing root

### Loading and Deflection Relations

The net wing beam load distribution along the span is given by

$$q(y) = L'(y) - N g m'(y) \quad (1)$$

where  $m'(y)$  is the local mass/span of the wing, and  $N$  is the load factor. In steady level flight we have  $N=1$ . The net loading  $q(y)$  produces shear  $\mathcal{S}(y)$  and bending moment  $M(y)$  in the beam structure. This resultant distribution produces a deflection angle  $\theta(y)$ , and deflection  $w(y)$  of the beam, as sketched in Figure 1.

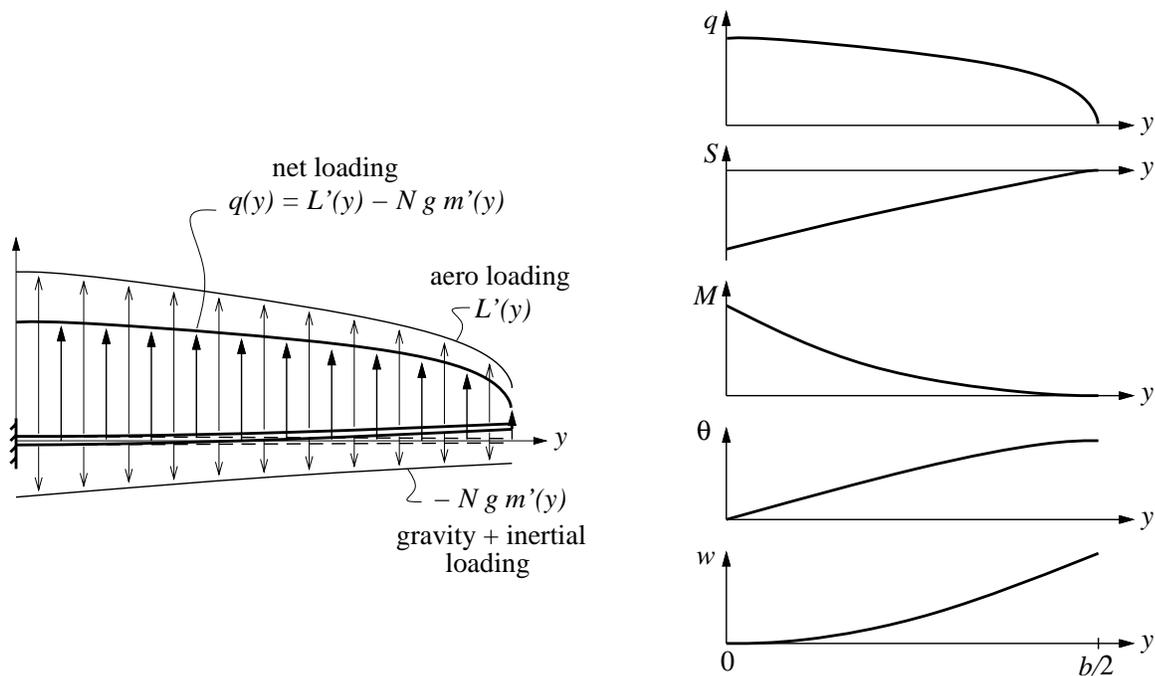


Figure 1: Aerodynamic and mass loadings, and resulting structural loads and deflection.

The standard differential equations derived via simple Bernoulli-Euler beam model, with the primary structural axis along the  $y$  direction, relate the loads and deflections to the loading  $q(y)$  and the bending stiffness  $EI(y)$ .

$$d\mathcal{S} = q dy \quad (2)$$

$$dM = \mathcal{S} dy \quad (3)$$

$$d\theta = \frac{M}{EI} dy \quad (4)$$

$$dw = \theta dy \quad (5)$$

To allow integration of these equations, it's necessary to impose four boundary conditions. For a cantilevered wing beam, they are

$$y = b/2 : \quad \mathcal{S} = 0 \quad (6)$$

$$y = b/2 : \quad M = 0 \quad (7)$$

$$y = 0 : \quad \theta = 0 \quad (8)$$

$$y = 0 : \quad w = 0 \quad (9)$$

### Load Distribution

The lift distribution  $L'(y)$  needed to define  $q(y)$  depends on the induced angle  $\alpha_i(y)$  and hence the overall wing shape in a complicated manner. One reasonable simplification is to assume that the net aerodynamic + weight loading in equation (1) is proportional to the local chord,

$$q(y) \simeq K_q c(y) \quad (10)$$

as shown in Figure 2.

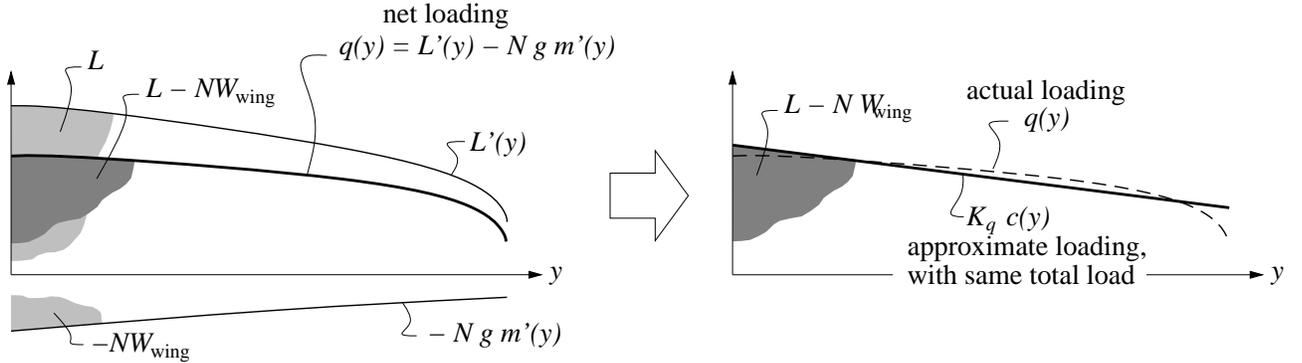


Figure 2: Net loading assumed to scale as local chord  $c(y)$ .

This is equivalent to assuming a constant local  $c_\ell = C_L$ , and that the local wing mass distribution  $m'(y)$  scales as the chord. The constant  $K_q$  is best set such that the approximate and actual loadings have the same total integrated loads.

$$\int_{-b/2}^{b/2} K_q c dy = \int_{-b/2}^{b/2} (L' - N g m') dy \quad (11)$$

$$K_q S_{\text{wing}} = L - N W_{\text{wing}} \quad (12)$$

$$K_q = \frac{L - N W_{\text{wing}}}{S_{\text{wing}}} = \frac{N W_{\text{fuse}}}{S_{\text{wing}}} \quad (13)$$

where  $W_{\text{fuse}}$  is the weight of the fuselage, defined as everything not including the wing.

### Numerical Integration

Equations (2) – (5) can be numerically integrated for any given nonuniform  $q(y)$  and  $EI(y)$  distributions. All spanwise variables are defined at a suitable number of discrete spanwise locations  $y_0, y_1 \dots y_i \dots y_{n-1}, y_n$ . The differential equations (2) – (5) above can then be approximated over the  $y_i \dots y_{i+1}$  interval via averages and finite differences.

$$\mathcal{S}_{i+1} - \mathcal{S}_i = \frac{q_{i+1} + q_i}{2} (y_{i+1} - y_i) \quad (14)$$

$$M_{i+1} - M_i = \frac{\mathcal{S}_{i+1} + \mathcal{S}_i}{2} (y_{i+1} - y_i) \quad (15)$$

$$\theta_{i+1} - \theta_i = \frac{1}{2} \left( \frac{M_{i+1}}{EI_{i+1}} + \frac{M_i}{EI_i} \right) (y_{i+1} - y_i) \quad (16)$$

$$w_{i+1} - w_i = \frac{\theta_{i+1} + \theta_i}{2} (y_{i+1} - y_i) \quad (17)$$

As indicated in Figure 3, this is equivalent to approximate integration via the Trapezoidal Rule.

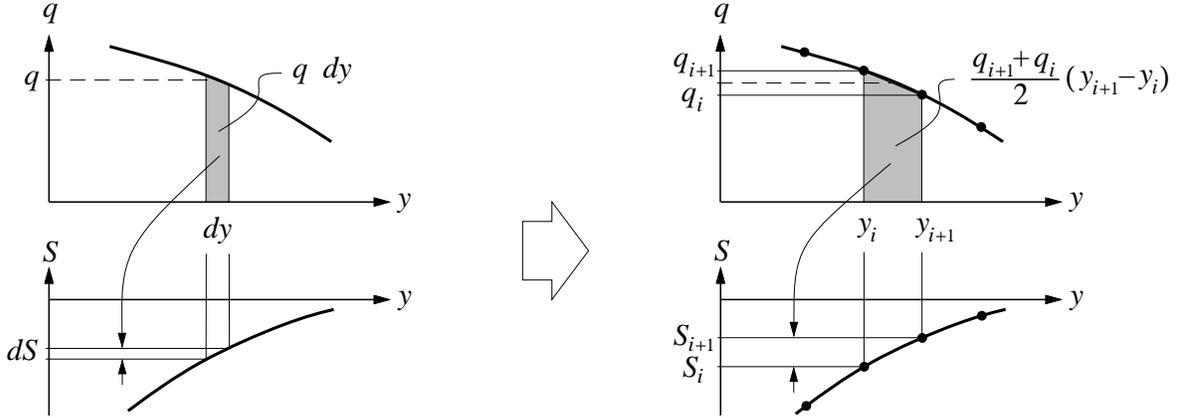


Figure 3: Approximation of continuous differential equation with discrete finite differences and averages (Trapezoidal Rule).

For this statically-determinate problem, the beam equations can be discretely integrated (i.e. summed) in the order written above. The summation of each equation starts at the end where its boundary condition is applied. Equations (14) and (15) have their boundary conditions at the tip at  $i = n$ , so they are summed from the tip inward after a simple rearrangement.

$$\mathcal{S}_n = 0 \quad (18)$$

$$M_n = 0 \quad (19)$$

$$\mathcal{S}_i = \mathcal{S}_{i+1} - \frac{q_{i+1} + q_i}{2} (y_{i+1} - y_i) \quad (i = n-1, n-2 \dots 0) \quad (20)$$

$$M_i = M_{i+1} - \frac{\mathcal{S}_{i+1} + \mathcal{S}_i}{2} (y_{i+1} - y_i) \quad (i = n-1, n-2 \dots 0) \quad (21)$$

For the two remaining equations (16) and (17), the boundary conditions (8) and (9) are at the root, so the summation proceeds from the root to the tip. After rearranging (16) and (17), and substituting  $i \rightarrow i-1$ , we have

$$\theta_0 = 0 \quad (22)$$

$$w_0 = 0 \quad (23)$$

$$\theta_i = \theta_{i-1} + \frac{1}{2} \left( \frac{M_i}{EI_i} + \frac{M_{i-1}}{EI_{i-1}} \right) (y_i - y_{i-1}) \quad (i = 1, 2 \dots n) \quad (24)$$

$$w_i = w_{i-1} + \frac{\theta_i + \theta_{i-1}}{2} (y_i - y_{i-1}) \quad (i = 1, 2 \dots n) \quad (25)$$

The summations above are readily implemented in a spreadsheet, as shown in Figure 4.

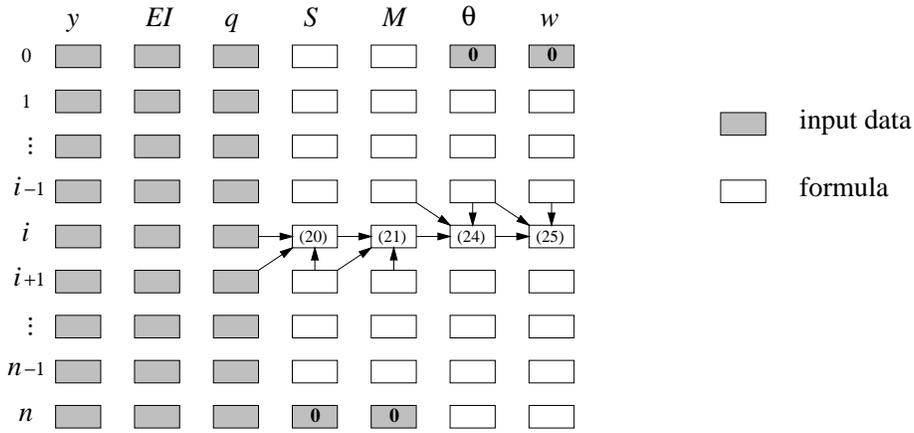


Figure 4: Spreadsheet computation of arbitrary cantilever beam deflection.

### Simplified Deflection Calculations

For preliminary or optimization work, the spreadsheet calculation of each candidate wing is unwieldy. For estimation, a very simple approximation is to assume that the beam curvature

$$\kappa(y) \equiv \frac{d^2w}{dy^2} = \frac{d\theta}{dy} = \frac{M(y)}{EI(y)}$$

is constant, and taken from some representative location such as the wing root at  $y=0$ .

$$\kappa(y) \simeq \kappa_0 = \frac{M_0}{EI_0} \quad (26)$$

$$\theta(y) = \int_0^y \kappa_0 dy = \kappa_0 y \quad (27)$$

$$w(y) = \int_0^y \theta dy = \frac{1}{2} \kappa_0 y^2 \quad (28)$$

For a straight-taper wing with taper ratio  $c_t/c_r = \lambda$ , the chord distribution is

$$c(y) = \frac{S_{\text{wing}}}{b} \frac{2}{1+\lambda} \left[ 1 + (\lambda - 1) \frac{2y}{b} \right] \quad (29)$$

and the corresponding approximate loading is then given by (10), and by the  $K_q$  definition (13).

$$q(y) \simeq K_q c(y) = \frac{NW_{\text{fuse}}}{b} \frac{2}{1+\lambda} [1 + (\lambda - 1)\eta] \quad (30)$$

$$\eta \equiv \frac{2y}{b} \quad (31)$$

To simplify the subsequent integrations, the  $y$  coordinate has been replaced with the equivalent and more convenient normalized coordinate  $\eta$ , which runs  $\eta = 0 \dots 1$  root to tip. The shear and bending moment are then calculated by integrating equations (2) and (3).

$$\mathcal{S}(y) = \int_y^{b/2} q(y) dy \quad (32)$$

$$\mathcal{S}(\eta) = \frac{b}{2} \int_\eta^1 q(\eta) d\eta \quad (33)$$

$$= \frac{NW_{\text{fuse}}}{b} \frac{2}{1+\lambda} \frac{b}{2} \int_\eta^1 [1 + (\lambda - 1)\eta] d\eta \quad (34)$$

$$= \frac{NW_{\text{fuse}}}{b} \frac{2}{1+\lambda} \frac{b}{2} \left[ 1 - \eta + (\lambda - 1) \frac{1}{2} (1 - \eta^2) \right] \quad (35)$$

$$M(y) = \int_y^{b/2} \mathcal{S}(y) dy \quad (36)$$

$$M(\eta) = \frac{b}{2} \int_\eta^1 \mathcal{S}(\eta) d\eta \quad (37)$$

$$= \frac{NW_{\text{fuse}}}{b} \frac{2}{1+\lambda} \frac{b^2}{4} \int_\eta^1 \left[ 1 - \eta + (\lambda - 1) \frac{1}{2} (1 - \eta^2) \right] d\eta \quad (38)$$

$$= \frac{NW_{\text{fuse}}}{b} \frac{2}{1+\lambda} \frac{b^2}{4} \left[ 1 - \eta - \frac{1}{2} (1 - \eta^2) + (\lambda - 1) \frac{1}{2} \left( 1 - \eta - \frac{1}{3} (1 - \eta^3) \right) \right] \quad (39)$$

The root moment is then

$$M_0 \equiv M(0) = NW_{\text{fuse}} \frac{b}{12} \frac{1+2\lambda}{1+\lambda} \quad (40)$$

which is subsequently combined with (26) and (28) to get the following estimate of the tip deflection.

$$\delta \equiv w(b/2) \simeq \frac{1}{2} \kappa_0 \left( \frac{b}{2} \right)^2 = \frac{M_0}{EI_0} \frac{b^2}{8} = \frac{NW_{\text{fuse}}}{EI_0} \frac{b^3}{96} \frac{1+2\lambda}{1+\lambda} \quad (41)$$

Figure 5 shows  $\kappa(y)$  for three taper ratios for a solid wing, for which the stiffness varies as  $EI(y) \sim c(y)^4$ . It can be seen that the  $\kappa = \kappa_0$  assumption is poor for a rectangular wing, but reasonable for wings of moderate to strong taper of  $\lambda = 0.3 \dots 0.5$ . Figure 6 compares the approximate deflections defined by (28) with the exact deflections, for the three taper ratios. For the untapered  $\lambda = 1.0$  wing, the approximation considerably overestimates the tip deflection, but the tapered cases are quite reasonable.

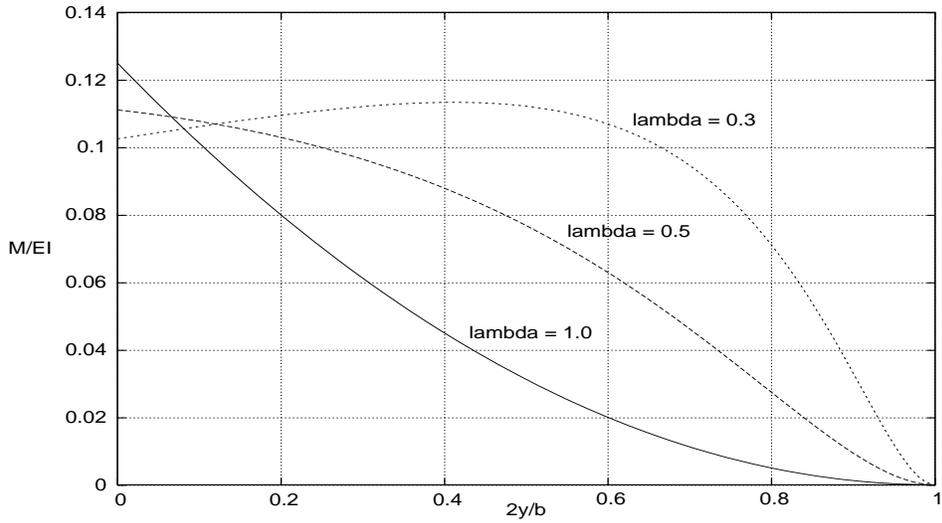


Figure 5: Spanwise distribution of curvature  $\kappa(y) = M/EI$  for three taper ratios.

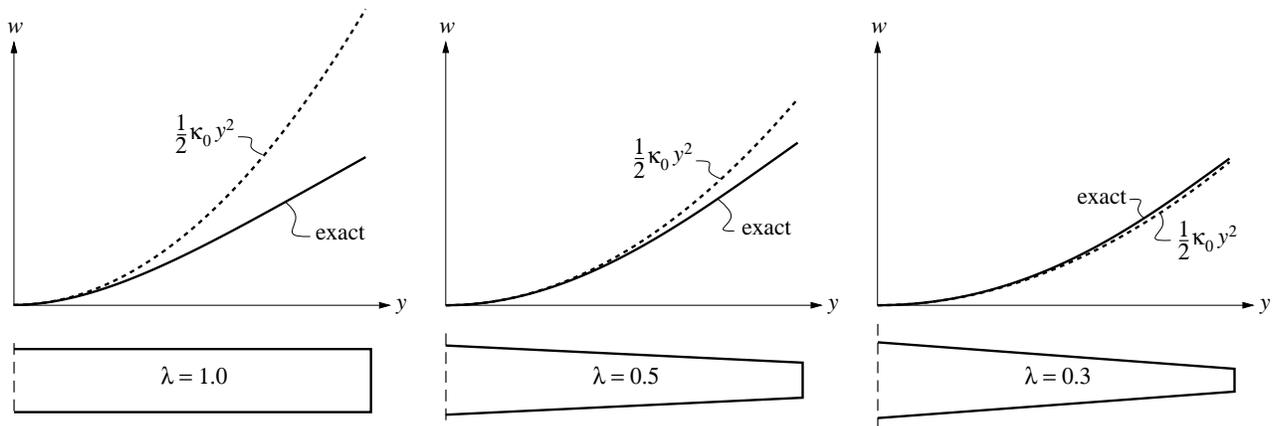


Figure 6: Approximate and exact deflections for three taper ratios.