Steps in Solving a Linear Dynamic Network Using Eigenvalue Method

1. Identify the states of the circuit (capacitor voltages and inductor currents), $\underline{x}$.

2. Use the node method (or loop method) to solve for unknown node voltages (or loop currents).

3. Voltage across inductors is difference in node potentials. (Current in capacitors is difference in loop currents.)

4. Find current in capacitors using KCL. (Find voltage across inductor using KVL.)

5. Using the constitutive relations for the capacitors and inductors, find the state equation of the form

$$\dot{\underline{x}} = A\underline{x}$$

6. Find the eigenvalues λ_i and the eigenvectors $\underline{v}_i$ of the matrix A.

7. The general solution is

$$\underline{x}(t) = \sum_i a_i \underline{v}_i e^{\lambda_i t}$$

8. Solve for the a_i that give the correct initial conditions. This may be calculated as

$$\underline{a} = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{bmatrix}^{-1} \underline{x}(0)$$

Lecture S13

From last time ...

$$\dot{\underline{x}} = \frac{d}{dt}\begin{pmatrix} v_1 \\ i_4 \end{pmatrix} = A\,\underline{x}$$

$$A = \begin{bmatrix} -1/2 & -2 \\ 1/2 & -3 \end{bmatrix}$$

The eigenvalues and eigenvectors are

$$\lambda_1 = -1; \quad \underline{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2.5; \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution is then

$$\underline{x}(t) = a_1 \underline{v}_1 e^{\lambda_1 t} + a_2 \underline{v}_2 e^{\lambda_2 t}$$

$$= a_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-t} + a_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2.5t}$$

The last step is to choose a_1, a_2 to match initial conditions.

$$x_1(0) = v_1(0) = 2 = 4a_1 + a_2$$

$$x_2(0) = i_4(0) = 1 = a_1 + a_2$$

Solving for a_1, a_2,

$$a_1 = 1/3$$
$$a_2 = 2/3$$

$$\Rightarrow \underline{x}(t) = \frac{1}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-t} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2.5t}$$

$$\Rightarrow \boxed{\begin{aligned} v_1(t) &= \frac{4}{3}e^{-t} + \frac{2}{3}e^{-2.5t} \\ i_4(t) &= \frac{1}{3}e^{-t} + \frac{2}{3}e^{-2.5t} \end{aligned}}$$

Of course, this is the same solution as before.

More generally,

$$\underline{x}(0) = a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_n\underline{v}_n$$

$$= \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

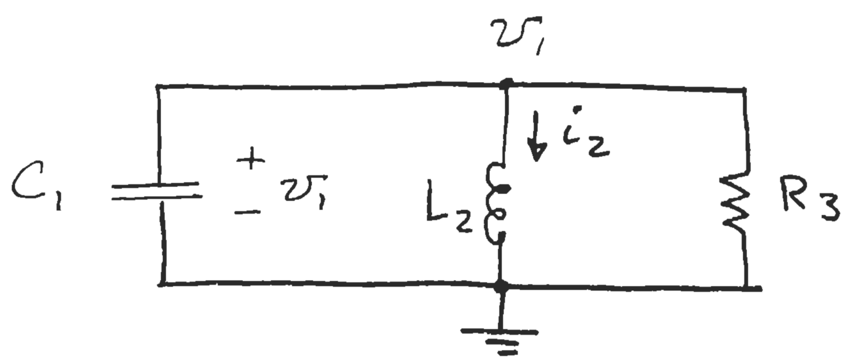
$$= V\underline{a}$$

So the initial condition is satisfied when

$$\underline{a} = V^{-1}\underline{x}(0)$$

Complex Eigenvalues and Eigenvectors

Do another example:



$C_1 = \frac{1}{2}\,F$

$L_2 = \frac{1}{5}\,H$

$R_3 = 1\,\Omega$

$v_1(0) = 1\,V$

$i_2(0) = 1\,V$

Identify states: $\underline{x} = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$

Find $\dot{\underline{x}}$: $\dfrac{dv_1}{dt} = \dfrac{1}{C_1} i_1$ ← need to find i_1

$\dfrac{di_2}{dt} = \dfrac{1}{L_2} v_2$ ← v_2 is a state — no further work needed

Find i_1:

KCL at v_1: $i_1 + i_2 + v_1/R_3 = 0$

$$\Rightarrow \quad i_1 = -\frac{v_1}{R_3} - i_2$$

So state equations are

$$\frac{d}{dt}\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{bmatrix} -1/R_3C_1 & -1/C_1 \\ 1/L_2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

or,

$$\dot{\underline{x}} = \begin{bmatrix} -2 & -2 \\ 5 & 0 \end{bmatrix} \underline{x}$$

The characteristic equation is

$$\phi(s) = |sI - A| = \begin{vmatrix} s+2 & +2 \\ -5 & s \end{vmatrix}$$

$$= (s+2)(s) - (-5)(2) = s^2 + 2s + 10 = 0$$

Use quadratic formula to find eigenvalues:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 10}}{2}$$

$$= -1 \pm 3j \quad \underline{\text{Complex!}}$$

$$\lambda_1 = -1 + 3j, \qquad \lambda_2 = -1 - 3j$$

Now find eigenvectors:

$$\lambda_1 I - A = \begin{bmatrix} 1+3j & 2 \\ -5 & -1+3j \end{bmatrix} = M$$

Row reduce:

$$\begin{array}{ccl} 1+3j & 2 & ① \\ \rightarrow\ 1 & \frac{1}{5} - \frac{3}{5}j & ①/(1+3j) = ③ \\ \hline -5 & -1+3j & ② \\ \rightarrow\ 0 & 0 & ④ = ② + 5① \end{array}$$

So reduced matrix is

$$M' = \begin{bmatrix} 1 & \frac{1}{5}(1-3j) \\ 0 & 0 \end{bmatrix}$$

So eigenvector is

$$\underline{v}_1 = \begin{bmatrix} -\frac{1}{5} + \frac{3}{5}j \\ 1 \end{bmatrix}$$

Note that $\lambda_2 = \lambda_1^*$ ← complex conjugate

$$\Rightarrow \underline{v}_2 = \underline{v}_1^* = \begin{bmatrix} -\frac{1}{5} - \frac{3}{5}j \\ 1 \end{bmatrix}$$

So the general solution is

$$\underline{x}(t) = \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

$$= a_1 \begin{bmatrix} -\frac{1}{5} + \frac{3}{5}j \\ 1 \end{bmatrix} e^{(-1+3j)t}$$

$$+ a_2 \begin{bmatrix} -\frac{1}{5} - \frac{3}{5}j \\ 1 \end{bmatrix} e^{(-1-3j)t}$$

Find a_1, a_2 to match ICs:

$$\underline{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -1/5 + 3/5\, j & -1/5 - \frac{3}{5}j \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= V \underline{a}$$

$$\underline{a} = V^{-1} \underline{x}(0) = \begin{pmatrix} 1/2 - 1j \\ 1/2 + 1j \end{pmatrix}$$

Therefore,

$$x(t) = \begin{bmatrix} 0.5 + 0.5j \\ 0.5 - j \end{bmatrix} e^{(-1+3j)t}$$

$$+ \begin{bmatrix} 0.5 - 0.5j \\ 0.5 + j \end{bmatrix} e^{(-1-3j)t}$$

To express in terms of real variables, use <u>Euler's Formula</u>:

$$e^{\alpha + j\beta} = e^{\alpha}(\cos\beta + j\sin\beta)$$

Therefore,

$$v_1(t) = (0.5+0.5j)(\cos 3t + j\sin 3t)e^{-t} + (0.5-0.5j)(\cos 3t - j\sin 3t)e^{-t}$$

$$= (\cos 3t - \sin 3t)\, e^{-t}$$

Likewise,

$$i_2(t) = (\cos 3t + 2\sin 3t)\, e^{-t}$$