# LECTURE S11

## Partial Fraction Expansions

In Lecture S10, we found the inverse Laplace transform "by inspection" — we recognized the ILT of

$$G(s) = \frac{b}{s-a}, \quad Re[s] > a$$

as $g(t) = b e^{-at} \sigma(t)$. <u>But</u>, we started off with LTs not in this form, e.g.,

$$G(s) = \frac{5s+12}{s^2+5s+6}$$

If we can express this $G(s)$ as a sum of first-order poles, we can easily find the inverse Laplace transform.

How can we do this? <u>Partial Fraction Expansion</u>

$$G(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

$$= N(s)/D(s)$$

Easiest case:

- $n > m$ ($G(s)$ is "strictly proper")
- $D(s) = (s-p_1)(s-p_2)\cdots(s-p_n)$ with <u>distinct</u> $p_i$ $[p_i \neq p_j]$

Then we can always write

$$G(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n}$$

Trick is to find $c_i$.

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<u>Example</u> $$G(s) = \frac{5s+12}{s^2+5s+6}$$

Find $p_i$ by solving $D(s) = 0$:

$$D(s) = s^2 + 5s + 6 = 0$$
$$\Longrightarrow s = -2, -3 \qquad (p_1, p_2)$$
$$\Longrightarrow D(s) = (s+2)(s+3)$$

So,

$$G(s) = \frac{c_1}{s+2} + \frac{c_2}{s+3}$$

How do we solve for $c_1, c_2$?

Bad way:

$$G(s) = \frac{c_1(s+3) + c_2(s+2)}{(s+3)(s+2)}$$

$$= \frac{(c_1+c_2)s + (3c_1+2c_2)}{s^2+5s+6}$$

$$= \frac{5s+12}{s^2+5s+6}$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 5 \\ 3c_1 + 2c_2 &= 12 \end{aligned}\right\} \text{ solve for } c_1, c_2$$

$$c_1 = \frac{\begin{vmatrix} 5 & 1 \\ 12 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{10-12}{2-3} = 2$$

$$c_2 = \frac{\begin{vmatrix} 1 & 5 \\ 3 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{12-15}{2-3} = 3$$

So $\quad G(s) = \dfrac{2}{s+2} + \dfrac{3}{s+3}$

## The Cover Up Method (the right way)

Note that

$$(s-p_1)G(s) = (s-p_1)\frac{c_1}{(s-p_1)} + (s-p_1)\frac{c_2}{(s-p_2)} + \cdots$$

At $s = p_1$,

$$\lim_{s \to p_1} (s-p_1)G(s) = \cancel{(s-p_1)}\frac{c_1}{\cancel{(s-p_1)}} + \overset{0}{\cancel{(s-p_1)}}\frac{c_2}{(s-p_2)} + \cdots$$

cancel

So, $$c_1 = \lim_{s \to p_1} (s-p_1)G(s)$$

and similarly for $c_2, c_3, \ldots$

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Example $$G(s) = \frac{5s+12}{s^2+5s+6} = \frac{5s+12}{(s+2)(s+3)}$$

$$c_1 = (s+2)G(s)\Big|_{s=-2} = \frac{5s+12}{s+3}\Big|_{s=-2}$$

$$= \frac{2}{1} = 2 \qquad \boxed{c_1 = 2}$$

$$C_2 = \frac{5s+12}{(s+2)\cancel{(s+3)}}\cancel{(s+3)}\Bigg|_{s=-3} = \frac{-3}{-1} = 3$$

$$\boxed{C_2 = 3}$$

just "cover up" this term

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Example Find the inverse LT of the unilateral LT

$$G(s) = \frac{1}{s^2+2s+5}$$

$$s^2+2s+5=0 \implies s = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm j \cdot 2$$

$$\implies G(s) = \frac{1}{[s-(-1+2j)][s-(-1-2j)]}$$

$$= \frac{c_1}{s-(-1+2j)} + \frac{c_2}{s-(-1-2j)}$$

Use cover up method to find $c_1, c_2$

$$c_1 = \frac{1}{-1+2j-(-1-2j)} = \frac{1}{4j}$$

$$C_2 = \frac{1}{-1-2j-(-1+2j)} = \frac{-1}{4j}$$

$$\Rightarrow \quad G(s) = \frac{1/4j}{s-(-1+2j)} - \frac{1/4j}{s-(-1-2j)}$$

Then the inverse LT is

$$g(t) = \frac{1}{4j} e^{(-1+2j)t} - \frac{1}{4j} e^{(-1-2j)t}$$

$$= \frac{1}{4j} e^{-t}\left(e^{2jt} - e^{-2jt}\right)$$

$$= \frac{1}{4j} e^{-t}\left(\left[\cos 2t + j\sin 2t\right] - \left[\cos 2t - j\sin 2t\right]\right)$$

$$= \frac{1}{2}\sin 2t \, e^{-t}, \qquad t \geq 0$$

$$g(t) = \begin{cases} \frac{1}{2}\sin 2t \, e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

This is easily verified, using LT tables.

<u>Note</u>: Method discussed today is general, except for cases with

- Repeated poles, or
- $m \geq n$ (as many "zeros" as "poles")

Will handle these cases next time.

## Partial Fraction Expansion (continued)

### Numerator Order ≥ Denominator Order

What happens when $m \geq n$?

$m = n$:

Example
$$G(s) = \frac{2s^2 + 6s + 5}{s^2 + 3s + 2}$$
$$= \frac{2s^2 + 6s + 5}{(s+1)(s+2)}$$

$G(s)$ can be expanded as

$$G(s) = C_0 + \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

What is $C_0$? Equations above must be true as $s \to \infty$.

$$\lim_{s \to \infty} G(s) = \frac{2}{1} = 2 \qquad (s^2 \gg s,\ s^2 \gg 1)$$

or, use L'Hôpital's rule. Also,

$$\lim_{s \to \infty} G(s) = C_0 + \frac{C_1}{\infty} + \frac{C_2}{\infty} = C_0$$

That is, the constant term is the ratio of the coefficient of the leading term in the numerator to the coefficient of the leading term in the denominator.

Remaing terms can be found by coverup method:

$$c_1 = \left.\frac{2s^2+6s+5}{s+2}\right|_{s=-1} = \frac{1}{1} = 1$$

$$c_2 = \left.\frac{2s^2+6s+5}{s+1}\right|_{s=-2} = \frac{1}{-1} = -1$$

$$\Rightarrow \quad G(s) = 2 + \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow \quad g(t) = 2\delta(t) + e^{-t}\sigma(t) - e^{-2t}\sigma(t)$$

$m > n$:

<u>Example</u> $\quad G(s) = \dfrac{2s^2+5s+5}{s+1}$

$$= c_0 s + c_1 + \frac{c_2}{s+1}$$

Find constants by long division:

$$
\begin{array}{r}
2s + 3 \\
s+1 \,\big)\, \overline{2s^2 + 5s + 5} \\
-(2s^2 + 2s) \\
\hline
3s + 5 \\
3s + 3 \\
\hline
2
\end{array}
\leftarrow \text{remainder.}
$$

$$\Rightarrow G(s) = 2s + 3 + \frac{2}{s+1}$$

[For higher order problem, would need to do coverup method on either $G(s)$ or remainder term]

$$\Rightarrow g(t) = 2\dot{\delta}(t) + 3\delta(t) + 2e^{-t}\sigma(t)$$

"doublet," which is really a differentiator.

# The Laplace Transform of a Convolution

Basic result:

$$\mathcal{L}[g(t) * u(t)] = G(s)U(s) = Y(s)$$

Two ways to show this:

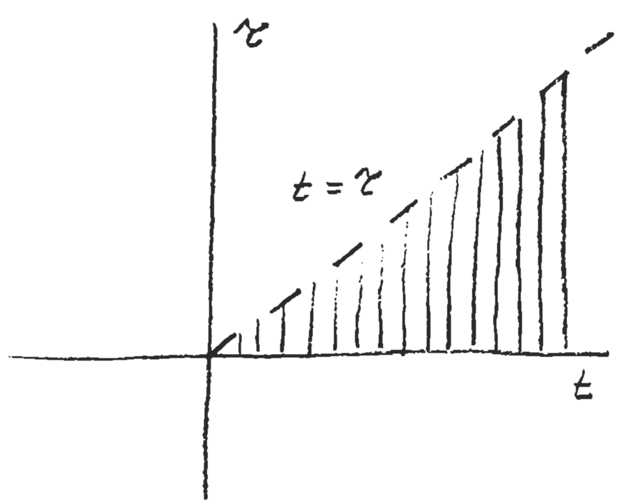
1. Direct Integration

$$Y(s) = \mathcal{L}[g(t) * u(t)]$$

$$= \int_0^\infty \left[\int_0^t g(t-\tau)u(\tau)\,d\tau\right] e^{-st}\,dt$$

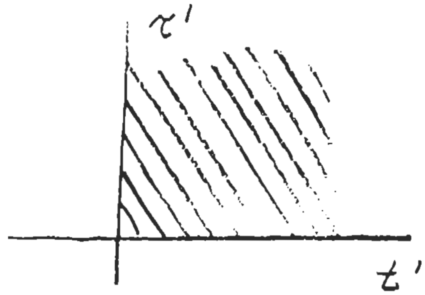
$$= \int_0^\infty \int_0^t g(t-\tau)u(\tau)e^{-st}\,d\tau\,dt$$

The region of integration is triangular:



Change variables to make integration area "square"

$$\left.\begin{aligned} t' &= t - \tau \\ \tau' &= \tau \end{aligned}\right\} \quad \begin{aligned} \tau &= \tau' \\ t &= t' + \tau' \end{aligned}$$



Change of variables in two dimensions:

$$\begin{pmatrix} t' \\ \tau' \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} t \\ \tau \end{pmatrix}$$

$$dt'\,d\tau' = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} dt\,d\tau = dt\,d\tau$$

(More generally, need the Jacobian of the transformation)

Therefore,

$$Y(s) = \int_0^\infty \int_0^\infty g(t')\,u(\tau')\,e^{-s(t'+\tau')}\,dt'\,d\tau'$$

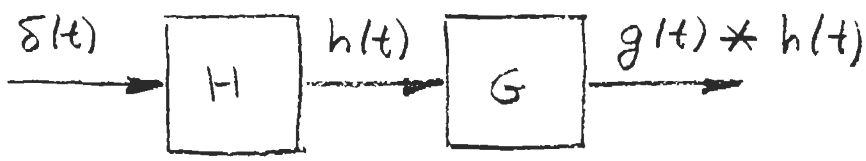
$$= \int_0^\infty \int_0^\infty g(t)\,u(\tau)\,e^{-st}\,e^{-s\tau}\,dt'\,d\tau'$$

$$= \int_0^\infty g(t)e^{-st}\,dt \int_0^\infty u(\tau)e^{-s\tau}\,d\tau$$

$$= G(s)\,U(s)$$

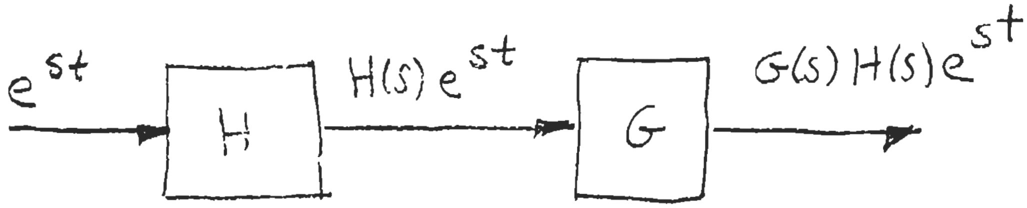
Note: the above derivation is valid so long as each integral is <u>absolutely</u> <u>convergent</u>

2. Use properties of linear systems:




So impulse response is $g(t) * h(t)$.

What is transfer function?

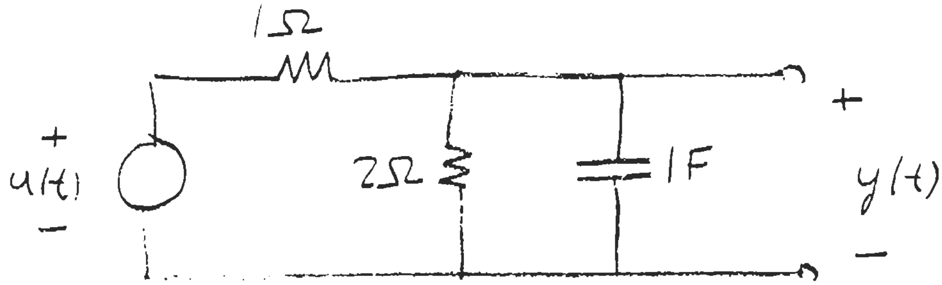



So transfer function is $G(s)\,H(s)$

$$\Rightarrow \mathcal{L}\left[g(t) * h(t)\right] = G(s)\,H(s).$$

Easy!

<u>Example</u> Find the response of the circuit




to the input $u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$G(s) =$ transfer function

$$= \frac{2 \| \frac{1}{s}}{2 \| \frac{1}{s} + 1} \quad \text{(voltage divider)}$$

$$2 \| 1/s = \frac{2/s}{2 + 1/s} = \frac{2}{2s+1}$$

$$G(s) = \frac{\frac{2}{2s+1}}{\frac{2}{2s+1} + 1} = \frac{2}{2 + 2s + 1}$$

$$= \frac{2}{2s+3} = \frac{1}{s+1.5}$$

$$U(s) = \mathcal{L}\left[u(t)\right] = \frac{1}{s+1}$$

$$Y(s) = G(s)\,U(s)$$

$$= \frac{1}{s+1.5}\,\frac{1}{s+1}$$

$$= \frac{2}{s+1} - \frac{2}{s+1.5}$$

$$\Rightarrow \quad y(t) = \left[ 2e^{-t} - 2e^{-1.5t} \right] \sigma(t)$$