Steps in solving a linear dynamic network (for now):

1. Choose to use node method or loop method, based on circuit topology and types of energy storage elements.
2. Write down node or loop equations, which yields differential and/or integral equations for variables.
3. Assume exponential solution, so that all variables are proportional to e^{st}.
4. Rewrite node or loop equations in terms of amplitudes.
5. Write equations in matrix form, e.g., $M(s)\underline{E} = \underline{0}$. N.B.: Can skip steps (2), (3), and (4), and do this step directly.
6. Find roots of equation $\det[M(s)] = 0$; for each root s_i, find $\underline{E}^i$.

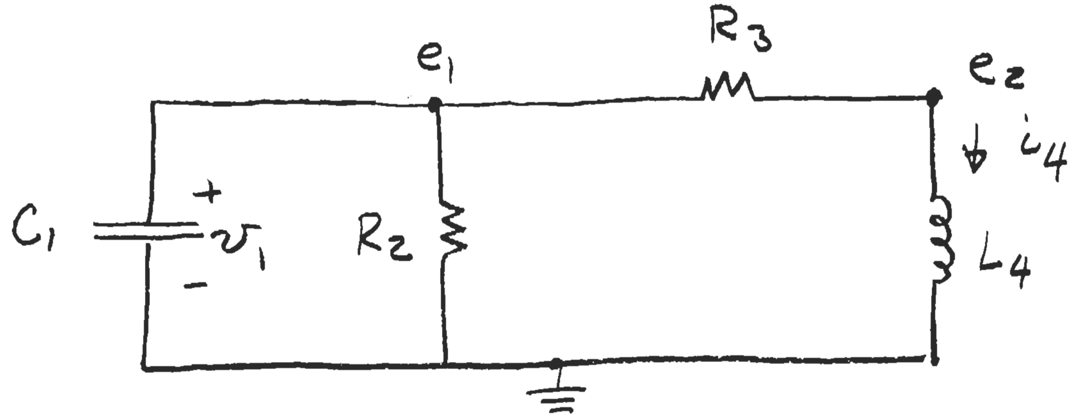
7. By superposition, general solution is

$$\underline{e}(t) = \sum_i a_i \underline{E}^i e^{s_i t}$$

8. From this solution, find the voltage or current in each capacitor or inductor.
9. Solve for the a_i that give the correct initial conditions.

Lecture S10

From last time:



$C_1 = \frac{1}{2} F \quad R_2 = 4\Omega \quad R_3 = 6\Omega \quad L_4 = 2H$

Using the node method, and assuming an exponential solution, we found

$$(C_1 s + G_2 + G_3) E_1 \qquad - G_3 E_2 = 0$$

$$- G_3 E_2 \quad + \left(G_3 + \frac{1}{L_4 s}\right) E_2 = 0$$

Note: $C_1 s, \frac{1}{L_4 s}$ are like conductances

They are called "admittances"

$L_4 s, \frac{1}{C_1 s}$ are like resistances

They are called "impedances"

Plug in component values:

$$\begin{bmatrix} \frac{1}{2}s + \frac{5}{12} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} + \frac{1}{2s} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M(s)\,\underline{E} = \underline{0}$$

The "characteristic equation" is

$$\phi(s) = \begin{vmatrix} \frac{1}{2}s + \frac{5}{12} & -1/6 \\ -1/6 & 1/6 + 1/2s \end{vmatrix}$$

$$= \frac{2s^2 + 7s + 5}{24s} = 0$$

The "characteristic values" are

$$s_1 = -1 \text{ sec}^{-1} \qquad s_2 = -5/2 \text{ sec}^{-1}$$

Solution procedure: For each characteristic value, find characteristic vector $\underline{E}$ that satisfies equations. Sum up solutions, and set constant such that initial conditions are satisfied.

$s_1 = -1$:

$$M(-1) = \begin{bmatrix} \frac{1}{2}(-1) + \frac{5}{12} & -1/6 \\ -1/6 & 1/6 + \frac{1}{2(-1)} \end{bmatrix}$$

$$= \begin{bmatrix} -1/12 & -1/6 \\ -1/6 & -1/3 \end{bmatrix}$$

Row reduce $M(-1)$

$$\begin{array}{cccl} 1 & -1/12 & -1/6 & \\ 3 & 1 & 2 & 1 \times (-12) \\ \hline 2 & -1/6 & -1/3 & \\ 4 & 0 & 0 & 2 + 1/6 \times 3 \end{array}$$

Reduced matrix is $\bar{M} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$$\bar{M}\,\underline{E} = \underline{0} \Rightarrow \underline{E} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So one solution is:

$$\underline{e}(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}$$

$s_2 = -5/2$:

$$M(-5/2) = \begin{bmatrix} -5/6 & -1/6 \\ -1/6 & -1/30 \end{bmatrix}$$

Row reduce:

①	$-5/6$	$-1/6$	
③	1	$1/5$	① $\times (-6/5)$
②	$-1/6$	$-1/30$	
④	0	0	② $+ \frac{1}{6} \times$ ③

Reduced matrix is $\bar{M} = \begin{bmatrix} 1 & 1/5 \\ 0 & 0 \end{bmatrix}$

$$\bar{M}\,\underline{E} = \underline{0} \implies \underline{E} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

So one solution is

$$\underline{e}(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-5/2\, t}$$

The general solution is

$$\underline{e}(t) = a\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + b\begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-\frac{5}{2}t}$$

Suppose the initial conditions are that

$$v_1(0) = 2 \text{ volt} \qquad i_4(0) = 1 \text{ ampere}$$

Can we use these to find a, b?
<u>Yes</u>, but it is difficult.

Express $v_1(t)$, $i_4(t)$ in terms of $e_1(t)$, $e_2(t)$:

$$v_1(t) = e_1(t) - 0 = e_1(t)$$

$$= -2ae^{-t} - be^{-5/2t}$$

$$i_4(t) = \frac{1}{L_4}\int v_4(t)\,dt$$

$$= \frac{1}{L_4}\int e_2(t)\,dt$$

$$= \frac{1}{2}\int\left(ae^{-t} + 5be^{-5/2t}\right)$$

$$= -\frac{a}{2}e^{-t} - be^{-5/2t}$$

Note: This step is a pain, especially if s_i are complex.

Therefore,

$$\left.\begin{array}{l} v_1(0) = -2a - b = 2 \\ i_4(0) = -\frac{a}{2} - b = 1 \end{array}\right\} \Rightarrow \begin{array}{l} a = -2/3 \\ b = -2/3 \end{array}$$

Therefore,

$$\boxed{\begin{array}{l} v_1(t) = \frac{4}{3}e^{-t} + \frac{2}{3}e^{-\frac{5}{2}t} \\ i_4(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{-\frac{5}{2}t} \end{array}}$$

Problems with approach:

1. Complicated, with many steps
2. Equations (in $\underline{e}(t)$) are not in terms of fundamental variables — $v_1(t)$ and $i_4(t)$

$\Rightarrow$ would like approach that emphasizes role of fundamental variables

The Concept of "State"

The state of a system is a set of variables (of smallest possible size) that, together with any inputs to the system, is sufficient to predict the future (important) behavior of the system.

- Each state variable is associated with "memory"
- Each state variable is associated with an initial condition
- State variables are often associated with energy storage.

In our example, the states are v_1, i_4, **NOT** e_1, e_2.

The *state vector* is the vector of state variables:

$$\underline{x} = \begin{bmatrix} v_1 \\ i_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The Definition of "State"

The *state* of a system is a set of variables (of smallest possible size) that, together with any inputs to the system, is sufficient to predict the future (important) behavior of the system.