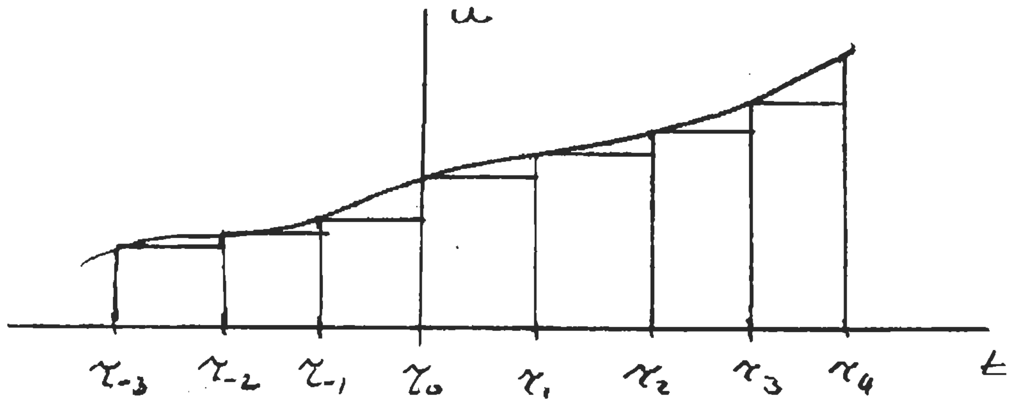
Lecture S4

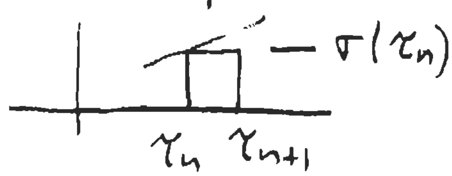
The Superposition Integral

Another way to look at the problems:



$u(t)$ is (approximately) a sum of pulses

$$u(t) \approx \sum_{n=-\infty}^{\infty} u(\tau_n) \underbrace{\left[\sigma(t-\tau_n) - \sigma(t-\tau_{n+1})\right]}_{\text{pulse}}$$



Then, by superposition,

$$y(t) \approx \sum_{n=-\infty}^{\infty} u(\tau_n) \left[g_s(t-\tau_n) - g_s(t-\tau_{n+1}) \right]$$

$$y(t) \approx \sum_{n=-\infty}^{\infty} u(\tau_n) \underbrace{\frac{\left[g_s(t-\tau_n) - g_s(t-\tau_{n+1})\right]}{\tau_{n+1} - \tau_n}}_{\approx g_s'(t-\tau_n) \,\equiv\, g(t-\tau_n)} \underbrace{(\tau_{n+1} - \tau_n)}_{\Delta\tau_n}$$

In the limit,

$$\boxed{y(t) = \int_{-\infty}^{\infty} g(t-\tau)\, u(\tau)\, d\tau}$$

This is the "Superposition Integral."
$g(t)$ is

$$g(t) = \frac{d}{dt} g_s(t)$$

$$= \text{"Impulse Response"}$$

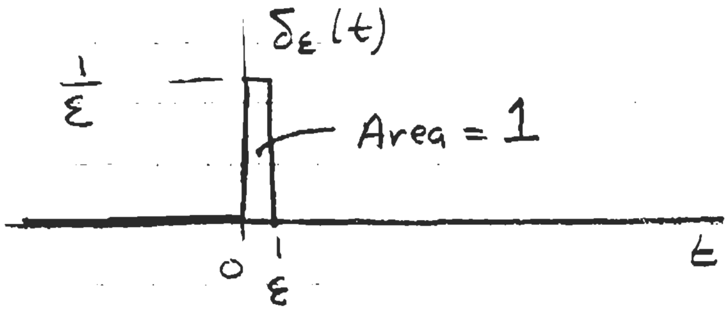
Why do we call it that?

The Impulse Function

A "pulse" is a short signal, often a square wave. The unit pulse function is

$$\delta_\varepsilon(t) = \begin{cases} 1/\varepsilon, & 0 \le t < \varepsilon \\ 0, & \text{else} \end{cases}$$

$$= \frac{1}{\varepsilon}\left[\sigma(t) - \sigma(t-\varepsilon)\right]$$



In the limit as $\varepsilon \to 0$, we get an "impulse":

$$\delta(t) \equiv \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

$$= \lim_{\varepsilon \to 0} \frac{\sigma(t) - \sigma(t-\varepsilon)}{\varepsilon}$$

$$= \frac{d}{dt}\sigma(t)$$

unit area
zero width

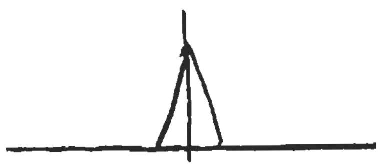
We have to be careful:

1) $\delta(t)$ is not really a function —

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

is not enough to define an impulse

2) We could have started with a pulse of <u>any</u> shape, and we would get the same result.

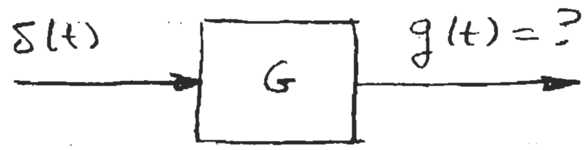


3) An impulse is really defined by what it <u>does</u>:

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau) = \sigma(t)$$

What is the response to an impulse?

The Impulse Response



$$\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\sigma(t) - \sigma(t-\varepsilon)\right]$$

$$\Rightarrow g(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[g_s(t) - g_s(t-\varepsilon)\right]$$

$$= \frac{d}{dt} g_s(t)$$

In words:

The <u>impulse response</u> is the derivative of the <u>step response</u>, because the <u>unit impulse</u> is the derivative of the <u>unit step</u>.

An Alternative Derivation of the Superposition Integral

Recall Duhamel's Integral:

$$y(t) = g_s(t)\,u(0) + \int_0^\infty g_s(t-\tau)\,u'(\tau)\,d\tau$$

This form is awkward, since we have to use derivative of $u(t)$. So integrate by parts:

$$\int U\,dV = UV - \int V\,dU$$

$$U = g_s(t-\tau) \implies dU = -g_s'(t-\tau)\,d\tau$$

$$dV = u'(\tau)\,d\tau \implies V = u(\tau)$$

Therefore,

$$y(t) = g_s(t)\,u(0) + g_s(t-\tau)\,u(\tau)\Big|_{\tau=0}^{\infty}$$

$$+ \int_0^\infty g_s'(t-\tau)\,u(\tau)\,d\tau$$

Recall $g(t) \equiv g_s'(t)$. Then

$$
\begin{aligned}
y(t) = {} & g_s(t)\,u(0) + g_s(-\infty)\cancelto{0}{u(\infty)} \quad \text{(because } g_s(-\infty) = 0\text{)} \\
& - g_s(t)\,u(0) \\
& + \int_0^\infty g(t-\tau)\,u(\tau)\,d\tau
\end{aligned}
$$

$$
\Rightarrow \quad y(t) = \int_0^\infty g(t-\tau)\,u(\tau)\,d\tau
$$

More generally, $u(\tau)$ need not be zero for $t < 0$. So,

$$
\boxed{y(t) = \int_{-\infty}^{\infty} g(t-\tau)\,u(\tau)\,d\tau}
$$