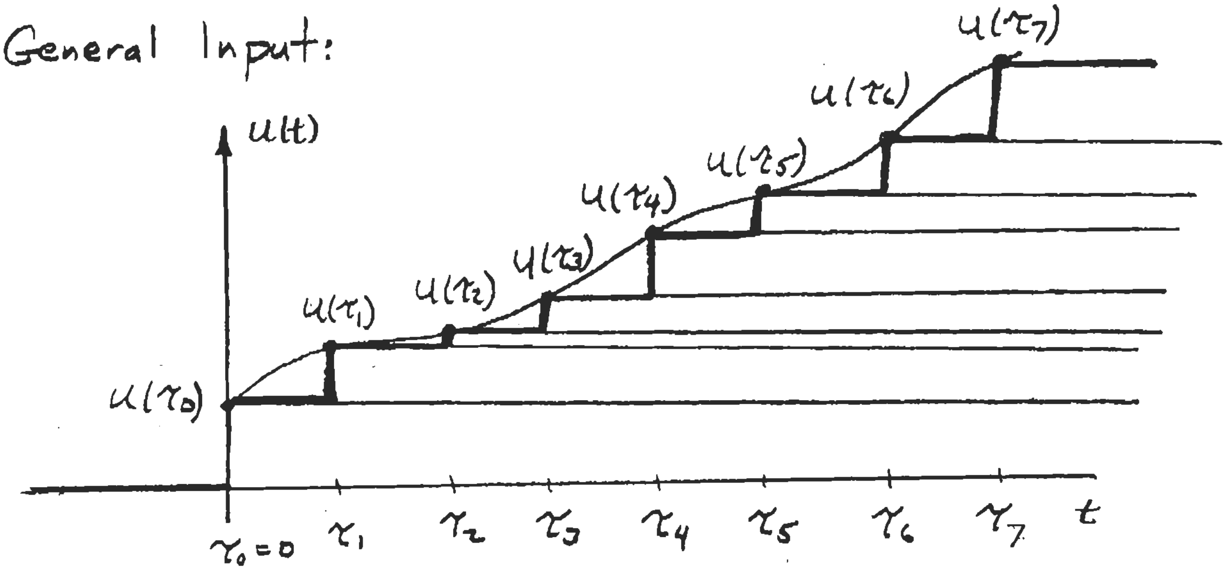
Lecture S3

Claim: If we know the step response of an LTI system, we can find the response to *any* input.

Approach:

- Approximate $u(t)$ by a staircase function
- Use superposition to find (approximate) $y(t)$.
- Take limit as staircase $\to u(t)$. Then output $\to y(t)$.

General Input:



Approximate expression for $u(t)$:

$$u(t) \approx u(\tau_0)\,\sigma(t-\tau_0) + \left[u(\tau_1) - u(\tau_0)\right]\sigma(t-\tau_1) + \left[u(\tau_2) - u(\tau_1)\right]\sigma(t-\tau_2) + \cdots$$

$$= u(\tau_0)\,\sigma(t-\tau_0) + \sum_{i=1}^{\infty}\left[u(\tau_i) - u(\tau_{i-1})\right]\sigma(t-\tau_i)$$

This is true whether system is LTI or not. But for LTI system, we can find response by superposition.

$$y(t) = \mathcal{G}[u(t)]$$

$$\approx u(\tau_0)\, g_s(t-\tau_0) + \sum_{i=1}^{\infty} \left[u(\tau_i) - u(\tau_{i-1})\right] g_s(t-\tau_i)$$

$$= u(\tau_0)\, g_s(t-\tau_0) + \sum_{i=1}^{\infty} \frac{u(\tau_i) - u(\tau_{i-1})}{\tau_i - \tau_{i-1}}\, g_s(t-\tau_i) \underbrace{\left[\tau_i - \tau_{i-1}\right]}_{d\tau}$$

In the limit as $\tau_i - \tau_{i-1} \to 0$, this becomes

$$y(t) = g_s(t-\tau_0)\, u(\tau_0) + \int_{\tau_0}^{\infty} g_s(t-\tau) \frac{du(\tau)}{d\tau}\, d\tau$$

Usually, $\tau_0 = 0$

$$\Rightarrow \quad y(t) = g_s(t)\, u(0) + \int_0^{\infty} g_s(t-\tau)\, u'(\tau)\, d\tau$$

(assumes $u(t) = 0,\ t < 0$)

$$= g_s(t)\, u(0) + \int_0^{t} g_s(t-\tau)\, u'(\tau)\, d\tau$$

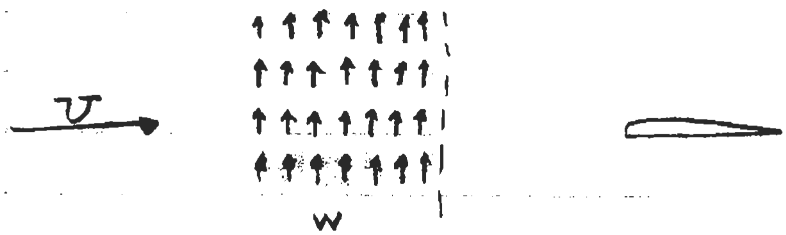
(if $g_s(t) = 0,\ t < 0$) "Causal" G

This is "Duhamel's superposition integral"

Note: will shortly put in a more useful form — Duhamel's integral not the usual one for S&S

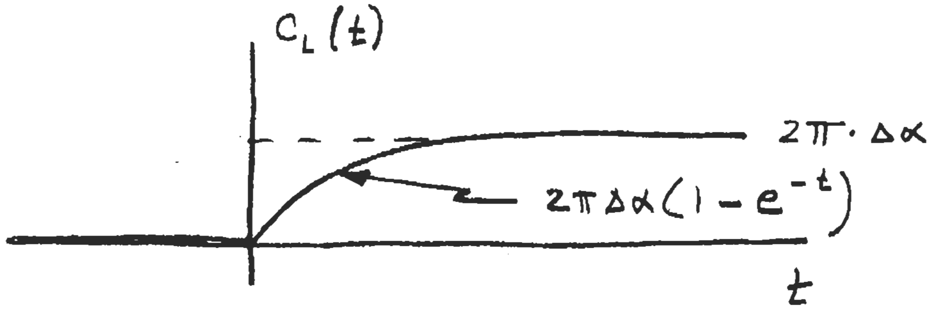
Duhamel's integral *is* often used in aerodynamics — often called the "indicial" response.

Example An airfoil in a wind tunnel is subjected to a "sharp-edged gust":



So the angle of attack, measured at the leading edge, is

For this change in angle of attack, the lift coefficient history is measured to be



[Note: real airfoils do this, but the form of the step response is more complicated]

So the response for a <u>unit</u> change in w is

$$g_s(t) = \begin{cases} 0, & t < 0 \\ \frac{2\pi}{v}\left(1-e^{-t}\right), & t \geq 0 \end{cases}$$

What is the response to a more gradual gust,

$$w(t) = \begin{cases} 0, & t < 0 \\ \overline{w}\left(1-e^{-2t}\right), & t \geq 0 \end{cases} \quad ?$$

Use Duhamel's integral:

causal response

$$C_L(t) = g_s(t)\, w(0) + \int_0^t g_s(t-\tau)\, w'(\tau)\, d\tau$$

$$w(0) = 0 \qquad w'(\tau) = \bar{w} \cdot 2e^{-2\tau}$$

$$\Rightarrow \quad C_L(t) = \int_0^t \frac{2\pi}{v}\left(1 - e^{-(t-\tau)}\right) \bar{w} \cdot 2e^{-2\tau}\, d\tau$$

$$= 4\pi \frac{\bar{w}}{v} \int_0^t \left[e^{-2\tau} - e^{-t-\tau}\right] d\tau$$

$$= \frac{4\pi \bar{w}}{v} \left[\frac{e^{-2\tau}}{-2} - \frac{e^{-t-\tau}}{-1}\right] \Bigg|_{\tau=0}^{t}$$

$$= \frac{4\pi \bar{w}}{v} \left[\frac{1}{2}\left(1 - e^{-2t}\right) + e^{-t}\left(e^{-t} - 1\right)\right]$$

$$= \frac{4\pi \bar{w}}{v} \left[\frac{1}{2} + \frac{1}{2} e^{-2t} - e^{-t}\right]$$

$$= \frac{2\pi \bar{w}}{v} \left[1 + e^{-2t} - 2e^{-t}\right], \quad t \geq 0$$