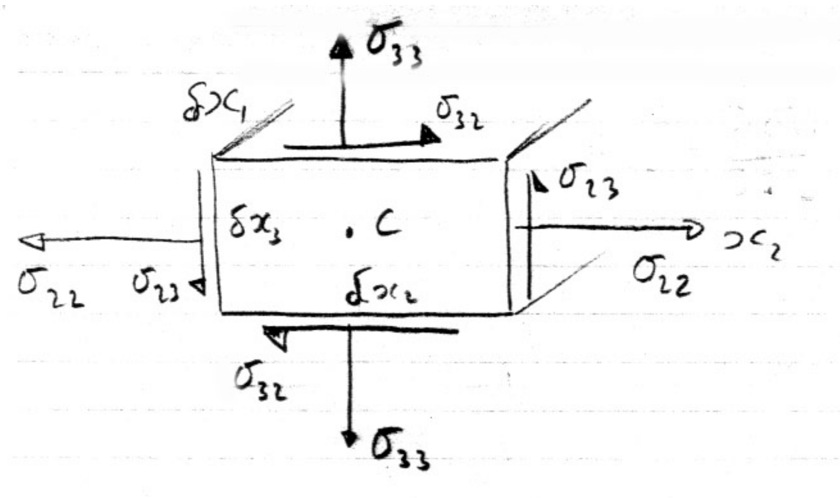
<u>Symmetry of Stress Tensor</u>

Consider moment equilibrium of differential element:



Taking moments about x_1 axis (i.e point C):

$$\sum M_1 = 0: \; 2\left[\sigma_{23}\underbrace{\left(\delta x_3 \delta x_1\right)}_{\text{Area of face}}\underbrace{\frac{\delta x_2}{2}}_{\text{Moment arm}}\right] - 2\left[\sigma_{32}\left(\delta x_2 \delta x_1\right)\frac{\delta x_3}{2}\right] = 0$$

$$\Rightarrow \sigma_{23} = \sigma_{32}$$

Thus, in general $\sigma_{mn} = \sigma_{nm}$

Stress tensor is symmetric. Six independent components of the stress tensor.

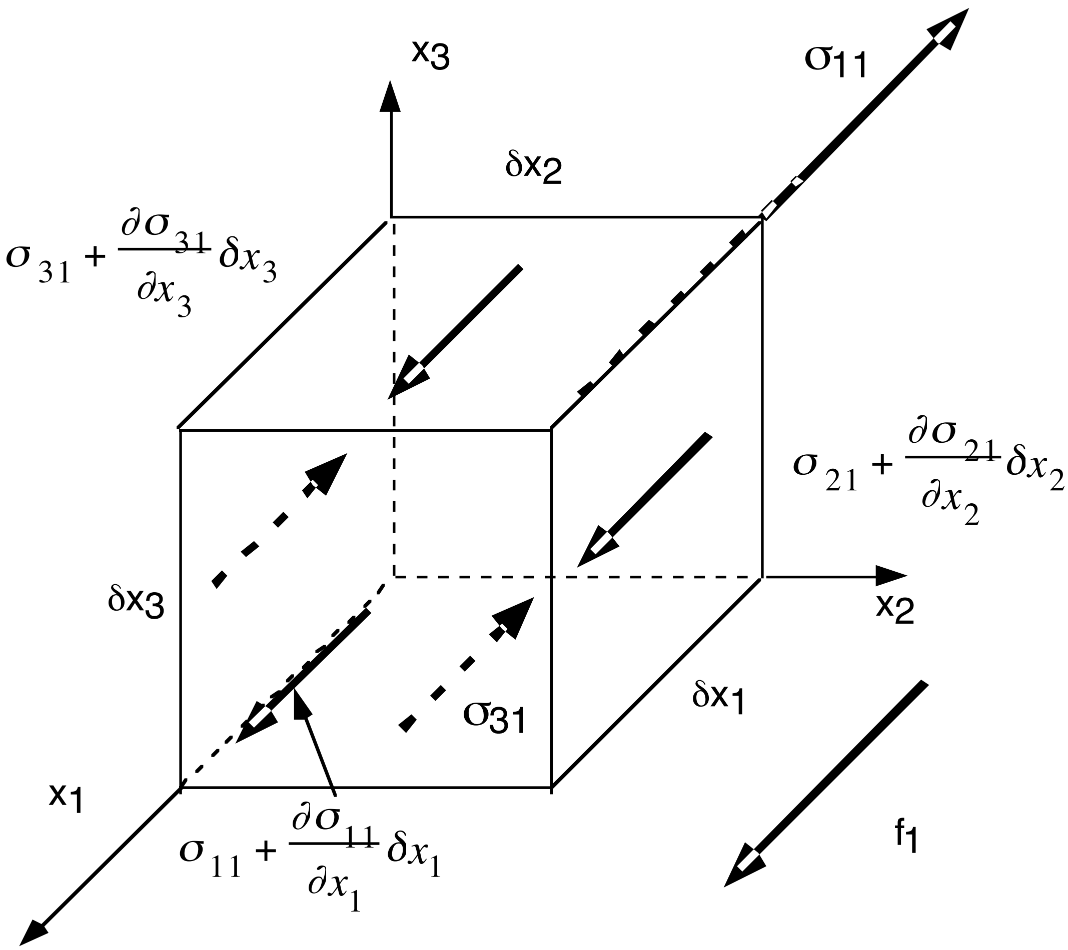
$$\begin{matrix} \sigma_{11} & \sigma_{12} & (=\sigma_{21}) \\ \sigma_{22} & \sigma_{23} & (=\sigma_{32}) \\ \sigma_{33} & \sigma_{31} & (=\sigma_{13}) \end{matrix}$$

Note a positive (tensile) component of stress acts on a face with a positive normal in a positive direction. Thus a stress acting on a negative normal face, in a negative direction is also positive. If the stresses do not vary over the infinitesimal element, σ_{mn} acts on opposite faces, in opposite directions but with equal magnitude.

But what happens if stress varies with position?

Stress Equations of Equilibrium

Consider equilibrium in x_1 direction. Let stresses vary across cube. Also allow there to be a body force (per unit volume) f_1 acting on the cube in the x1 direction:



f_n is a "body force", e.g. due to the weight of the element (ρg), centrifugal acceleration ($\rho r \omega^2$), electromagnetic fields etc.

Taking equilibrium of forces in the x_1 direction gives:

$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\delta x_1\right)(\delta x_2 \delta x_3) - \sigma_{11}(\delta x_2 \delta x_3) + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\delta x_2\right)(\delta x_1 \delta x_3) - \sigma_{21}(\delta x_1 \delta x_3)$$
$$+\left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3}\delta x_3\right)(\delta x_1 \delta x_2) - \sigma_{31}(\delta x_1 \delta x_2) + f_1(\delta x_1 \delta x_2 \delta x_3) = 0$$

which simplifies to:

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} + f_1 = 0$$

similarly, equilibrium in the x_2 and x_3 directions yields:

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3} + f_2 = 0 \quad \text{and} \qquad \frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + f_3 = 0$$

Canceling out terms and dividing through by the common $dx_1, \quad dx_2, \quad dx_3$.

Gives:

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} + f_1 = 0 \qquad x_1 \text{ direction}$$

Similarly,

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_2} + f_2 = 0 \qquad x_2 \text{ direction}$$

$$\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + f_3 = 0 \qquad x_3 \text{ direction}$$

These are the three equations of stress equilibrium.

In tensor form:

$$\frac{\partial\sigma_{mn}}{\partial x_m} + f_n + 0$$

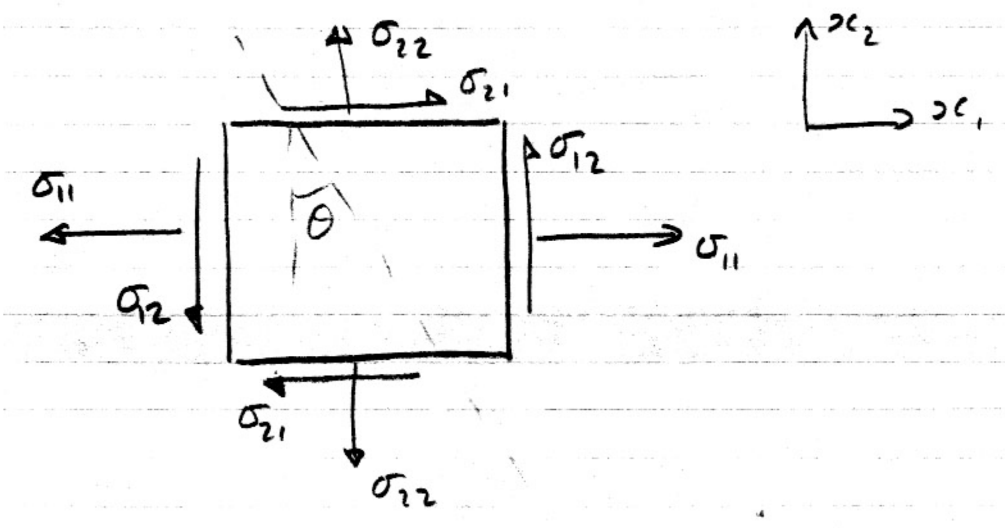
<u>Stress Transformations</u>

Just as we need to resolve forces with respect to structural axes - need to resolve stress.

- E.g. to identify directions which transmit maximum stress (tensile, shear, compressive)

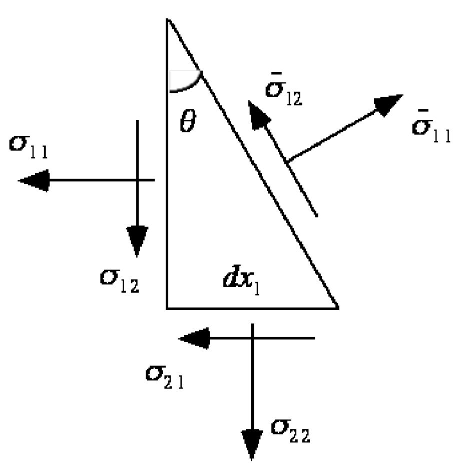
Two Dimensional Case:

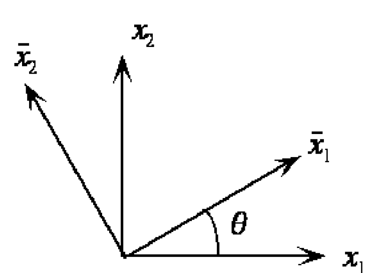
Consider a unit square (of unit depth dx_3)



What is the stress acting at some angle θ?

Consider equilibrium of a triangular, wedge shaped element:





- Align $\tilde{x}_1$ perpendicular to cut face, $\tilde{x}_2$ parallel to cut face

- The element is of uniform depth dx_3

- Must still be in equilibrium of forces:

$$\sum F = 0$$

Σ(Stress component x length x depth)

note: $dx_2 = d\tilde{x}_2 \cos\theta$

$dx_1 = d\tilde{x}_2 \sin\theta$

Equation of equilibrium in $\tilde{x}_1$ direction:

area

Resolved component

$$\tilde{\sigma}_{11} dx_3 d\tilde{x}_2 - \sigma_{11} dx_3 \left[\delta\tilde{x}_2 \cos\theta\right] \cos\theta - \sigma_{22} dx_3 \left[\delta\tilde{x}_2 \sin\theta\right] \sin\theta - \sigma_{12} dx_3 \left[\delta\tilde{x}_2 \sin\theta\right] \cos\theta - \sigma_{21} dx_3 \left[\delta\tilde{x}_2 \cos\theta\right] \sin\theta$$

canceling out $\delta\tilde{x}_2$ and combining

$$\tilde{\sigma}_{11} = \cos^2\theta\, \sigma_{11} + \sin^2\theta\, \sigma_{22} + 2\cos\theta \sin\theta\, \sigma_{12}$$

Note: 2 angles: 1 for area; one for resulting force

This can be done for $\tilde{\sigma}_{12}$ *and* $\tilde{\sigma}_{22}$, e.g.

$$\tilde{\sigma}_{12} = -\sigma_{11} \cos\theta \sin\theta + \sigma_{22} \sin\theta \cos\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right)$$

note you can always take equilibrium of an element

Tensor form

Transformation requires direction cosines (see below)

Stresses are second order tensors (2 subscripts) and require two direction cosines for transformation.

Thus:

$\tilde{\sigma}_{mn} = \ell_{\tilde{m}p}\ell_{\tilde{n}q}\sigma_{pq}$

- 2nd order tensor requires 2 directional cosines
- transforms stress from x_n system to $\tilde{x}_m$ system

one angle for area, one for component of force

In 2-D $\tilde{\sigma}_{\gamma\beta} = \ell_{\tilde{\gamma}\theta}\ell_{\tilde{\beta}\tau}\sigma_{\theta\tau}$

Side Note: Direction Cosines

Define direction cosine as:

Angle between

$\ell_{m\tilde{n}} = \cos\left[x_m \tilde{x}_n\right]$

such that

$x_m = \ell_{m\tilde{n}}\ x_{\tilde{n}}$

$or\ \tilde{x}_n = \ell_{\tilde{n}m}\ x_m$

} Tensor equations

where we mean

$x_1 = \ell_{1\tilde{1}}\tilde{x}_{\tilde{1}} + \ell_{1\tilde{2}}\tilde{x}_{\tilde{2}} + \ell_{1\tilde{3}}\tilde{x}_{\tilde{3}}$

$x_2 = \ell_{2\tilde{n}}\tilde{x}_{\tilde{n}}$

$x_3 = \ell_{3\tilde{n}}\tilde{x}_{\tilde{n}}$

We could who use the same idea for forces

$\tilde{F}_n = \ell_{\tilde{n}m}F_m$

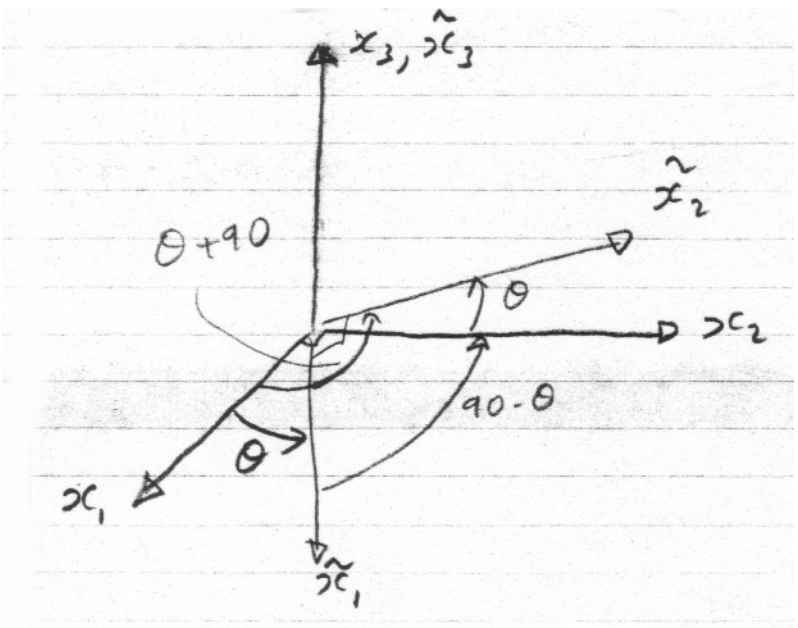
First order tensor is a vector (one subscript) requires one direction cosine for transformation.

Angles measured positive counter-clockwise.

$\ell_{\tilde{m}n} = \ell_{n\tilde{m}}$ Since cosine is an even function $\cos(\theta) = \cos(-\theta)$

But $\ell_{\tilde{m}n} \quad \ell_{\tilde{n}m}$ - different angles

2-D Example



	x_1	x_2	x_3
$\tilde{x}_1$	$\ell_{\tilde{1}1} = \cos\theta$	$\ell_{\tilde{1}2} = \cos(90-\theta) = \sin\theta$	$\ell_{\tilde{1}3} = \cos 90 = 0$
$\tilde{x}_2$	$\ell_{\tilde{2}1} = \cos(-(90+\theta))$	$\ell_{\tilde{2}2} = \cos\theta$	$\ell_{\tilde{2}3} = \cos 90 = 0$
$\tilde{x}_3$	$\ell_{\tilde{3}1} = \cos 90 = 0$	$\ell_{\tilde{3}2} = \cos 90 = 0$	$\ell_{\tilde{3}3} = \cos\theta = 1$

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One further way to transform stress is to use "Mohr's Circle" graphical method for rotating 2[nd] order tensors in 2-D coordinate system

Mohr's Circle for Stress

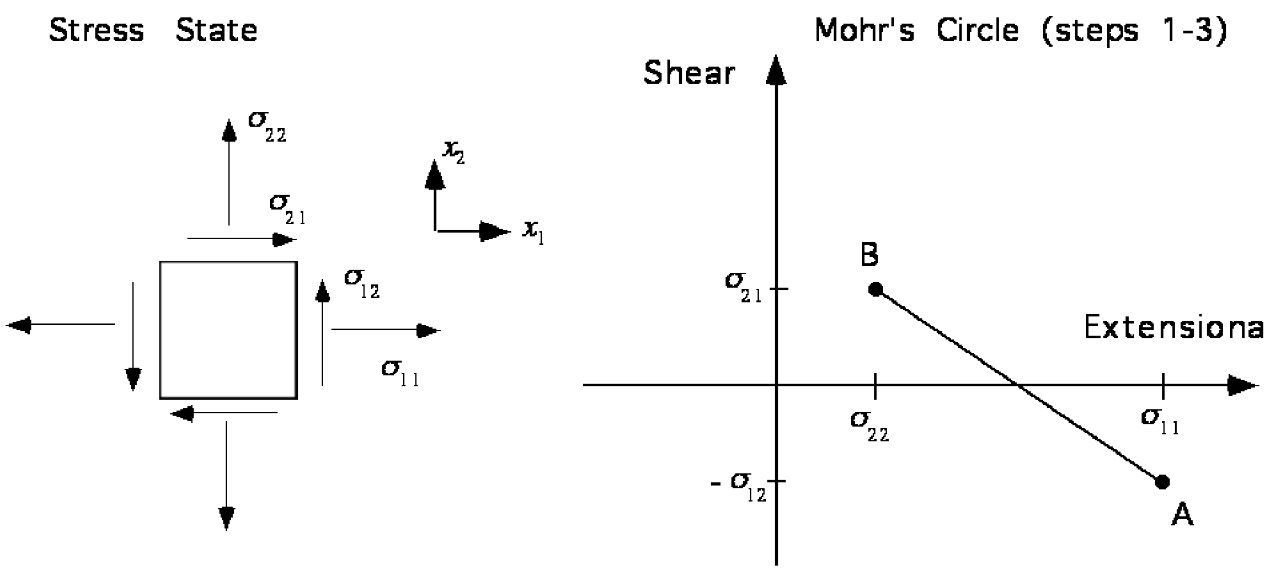
Mohr's circle is a geometric representation of the 2-D transformation of stresses.

Construction: Given the state of stress shown below, with the following definition (by Mohr) of positive and negative shear:

"Positive shear would cause a clockwise rotation of the infinitessimal element about the element center."

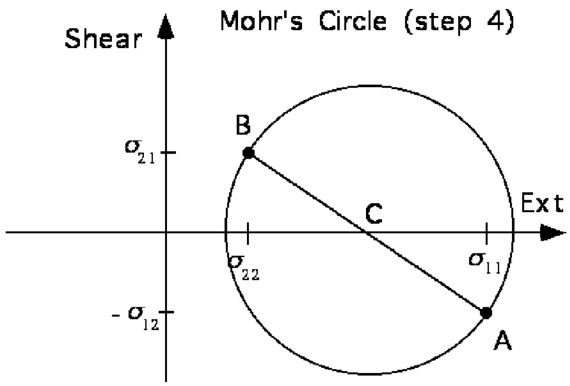
Thus: σ_{21} (*below*) is plotted positive

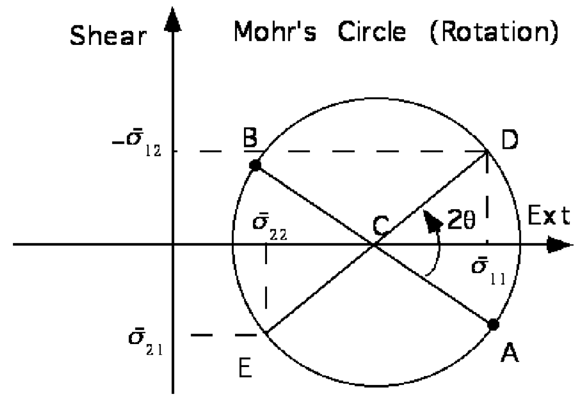
σ_{12} (*below*) is plotted negative:



Following these rules do the following:

1. Plot σ_{11}, $-\sigma_{12}$ as point A
2. Plot σ_{22}, σ_{21} as point B
3. Connect A and B.
4. Draw a circle of diameter of the line AB about the point where the line AB crosses the horizontal axis (denote this as C).





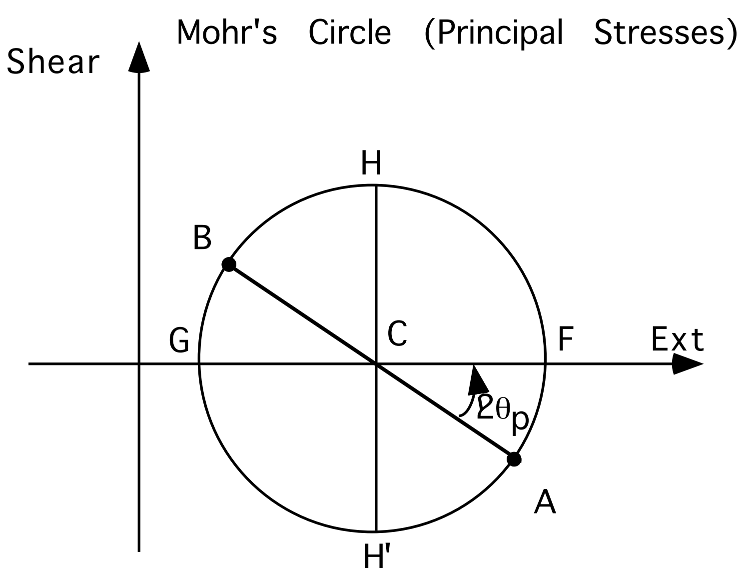
To read off the stresses for a rotated system:

1. Note that the vertical axis is the shear axis; the horizontal axis is the extensional stress axis.

2. Positive rotations are measured counter clockwise as referenced to the original system and thus to the line AB.

3. Rotate line AB about point C by the angle 2θ, where θ is the angle between the unrotated and rotated system.

4. The points D and E where the rotated line intersects the circle are used to read off the stresses in the rotated system. The vertical location of D is $-\tilde{\sigma}_{12}$; the horizontal location of D is $\tilde{\sigma}_{11}$. The vertical location of E is $\tilde{\sigma}_{21}$, the horizontal location of E is $\tilde{\sigma}_{22}$



We can also immediately see the following:

1. The principal stresses (see below), σ_I and σ_{II} are defined by the points G and F (along the horizontal axis $\tilde{\sigma}_{12} = 0$). The rotation angle to the principal axis is θ_p which is $\frac{1}{2}$ the angle from the line AB to the horizontal line FG.

2. The maximum shear stress is defined by the points H and H' which are the endpoints of the vertical line. The line is orthogonal to the principal stress line and thus the maximum shear stress acts along a plane 45° (= 90°/2) from the principal stress system.

Principal Stresses

The principal Stresses are the *extremum* (maximum or minimum) values of stress at a point. The principal directions are the directions corresponding to these *extremum* values of stress. The principal directions have no shear stresses associated with them (consider differential equation for stress equilibrium).