

# Fluids – Lecture 15 Notes

## 1. Uniform flow, Sources, Sinks, Doublets

Reading: Anderson 3.9 – 3.12

### Uniform Flow

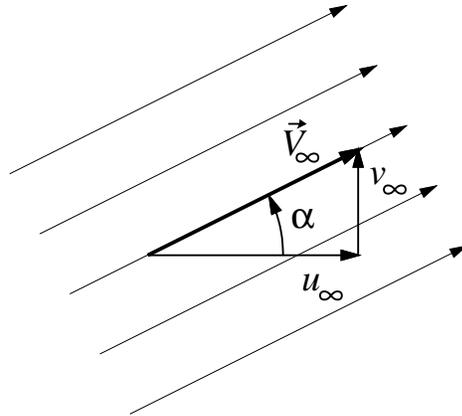
#### Definition

A *uniform flow* consists of a velocity field where  $\vec{V} = u\hat{i} + v\hat{j}$  is a constant. In 2-D, this velocity field is specified either by the freestream velocity components  $u_\infty$ ,  $v_\infty$ , or by the freestream speed  $V_\infty$  and flow angle  $\alpha$ .

$$\begin{aligned}u &= u_\infty = V_\infty \cos \alpha \\v &= v_\infty = V_\infty \sin \alpha\end{aligned}$$

Note also that  $V_\infty^2 = u_\infty^2 + v_\infty^2$ . The corresponding potential and stream functions are

$$\begin{aligned}\phi(x, y) &= u_\infty x + v_\infty y = V_\infty (x \cos \alpha + y \sin \alpha) \\ \psi(x, y) &= u_\infty y - v_\infty x = V_\infty (y \cos \alpha - x \sin \alpha)\end{aligned}$$



#### Zero Divergence

A uniform flow is easily shown to have *zero divergence*

$$\nabla \cdot \vec{V} = \frac{\partial u_\infty}{\partial x} + \frac{\partial v_\infty}{\partial y} = 0$$

since both  $u_\infty$  and  $v_\infty$  are constants. The equivalent statement is that  $\phi(x, y)$  satisfies Laplace's equation.

$$\nabla^2 \phi = \frac{\partial^2 (u_\infty x + v_\infty y)}{\partial x^2} + \frac{\partial^2 (u_\infty x + v_\infty y)}{\partial y^2} = 0$$

Therefore, the uniform flow satisfies mass conservation.

#### Zero Curl

A uniform flow is also easily shown to be irrotational, or to have *zero vorticity*.

$$\nabla \times \vec{V} \equiv \vec{\xi} = \left( \frac{\partial v_\infty}{\partial x} - \frac{\partial u_\infty}{\partial y} \right) \hat{k} = 0$$

The equivalent irrotationality condition is that  $\psi(x, y)$  satisfies Laplace's equation.

$$\nabla^2 \psi = \frac{\partial^2(u_\infty y - v_\infty x)}{\partial x^2} + \frac{\partial^2(u_\infty y - v_\infty x)}{\partial y^2} = 0$$

## Source and Sink

### Definition

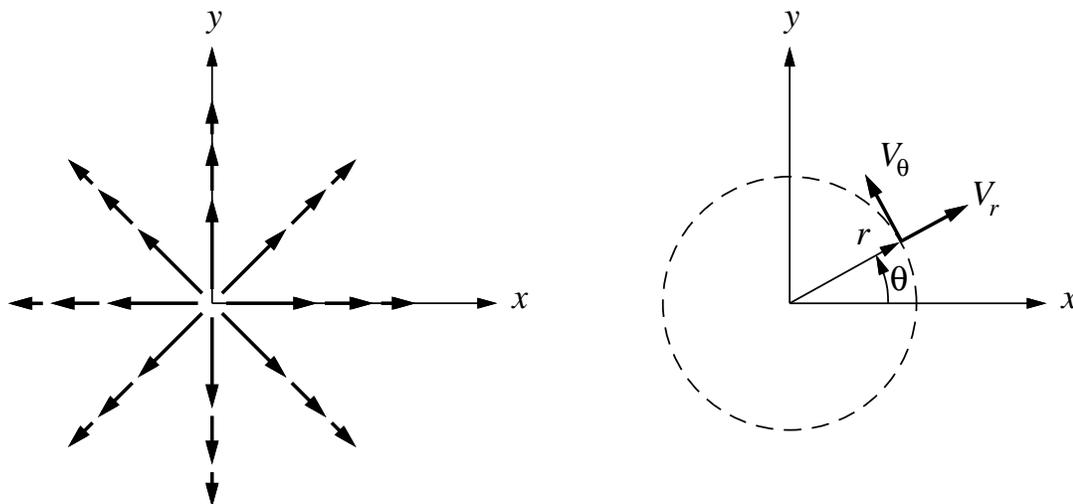
A 2-D *source* is most clearly specified in polar coordinates. The radial and tangential velocity components are defined to be

$$V_r = \frac{\Lambda}{2\pi r} \quad , \quad V_\theta = 0$$

where  $\Lambda$  is a scaling constant called the *source strength*. The volume flow rate per unit span  $\dot{\mathcal{V}}'$  across a circle of radius  $r$  is computed as follows.

$$\dot{\mathcal{V}}' = \int_0^{2\pi} \vec{V} \cdot \hat{n} dA = \int_0^{2\pi} V_r r d\theta = \int_0^{2\pi} \frac{\Lambda}{2\pi r} r d\theta = \Lambda$$

Hence we see that the source strength  $\Lambda$  specifies the rate of volume flow issuing outward from the source. If  $\Lambda$  is negative, the flow is inward, and the flow is called a *sink*.



### Cartesian representation

The cartesian velocity components of the source or sink are

$$u(x, y) = \frac{\Lambda}{2\pi} \frac{x}{x^2 + y^2}$$

$$v(x, y) = \frac{\Lambda}{2\pi} \frac{y}{x^2 + y^2}$$

and the corresponding potential and stream functions are as follows.

$$\phi(x, y) = \frac{\Lambda}{2\pi} \ln \sqrt{x^2 + y^2} = \frac{\Lambda}{2\pi} \ln r$$

$$\psi(x, y) = \frac{\Lambda}{2\pi} \arctan(y/x) = \frac{\Lambda}{2\pi} \theta$$

It is easily verified that apart from the origin location  $(x, y) = (0, 0)$ , these functions satisfy  $\nabla^2\phi = 0$  and  $\nabla^2\psi = 0$ , and hence represent physically-possible incompressible, irrotational flows.

### Singularities

The origin location  $(0, 0)$  is called a *singular point* of the source flow. As we approach this point, the magnitude of the radial velocity tends to infinity as

$$V_r \sim \frac{1}{r}$$

Hence the flow at the singular point is not physical, although this does not prevent us from using the source to represent actual flows. We will simply need to ensure that the singular point is located outside the flow region of interest.

### Uniform Flow with Source

Two or more incompressible, irrotational flows can be combined by *superposition*, simply by adding their velocity fields or their potential or stream function fields. Superposition of a uniform flow in the  $x$ -direction and a source at the origin therefore has

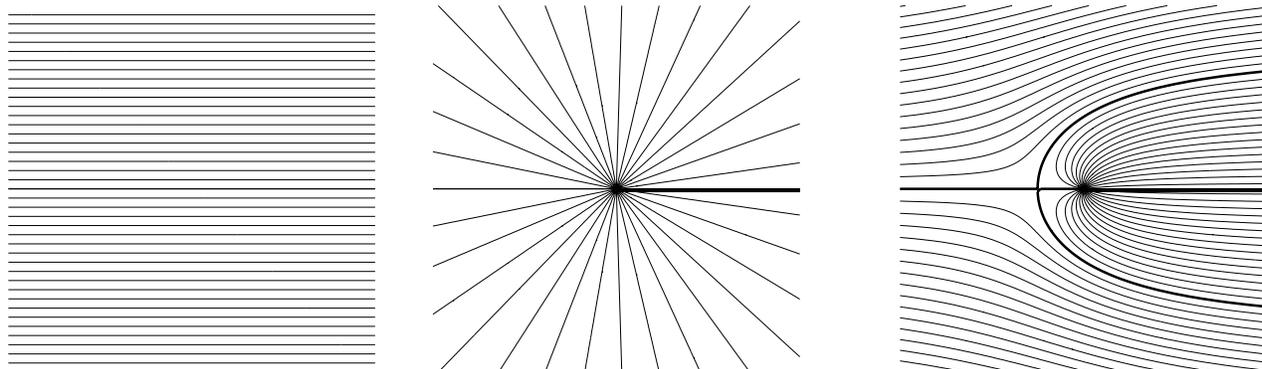
$$u(x, y) = \frac{\Lambda}{2\pi} \frac{x}{x^2 + y^2} + V_\infty$$

$$v(x, y) = \frac{\Lambda}{2\pi} \frac{y}{x^2 + y^2}$$

or 
$$\phi(x, y) = \frac{\Lambda}{2\pi} \ln \sqrt{x^2 + y^2} + V_\infty x = \frac{\Lambda}{2\pi} \ln r + V_\infty r \cos \theta$$

or 
$$\psi(x, y) = \frac{\Lambda}{2\pi} \arctan(y/x) + V_\infty y = \frac{\Lambda}{2\pi} \theta + V_\infty r \sin \theta$$

The figure shows the streamlines of the two basic flows, and also the combined flow.

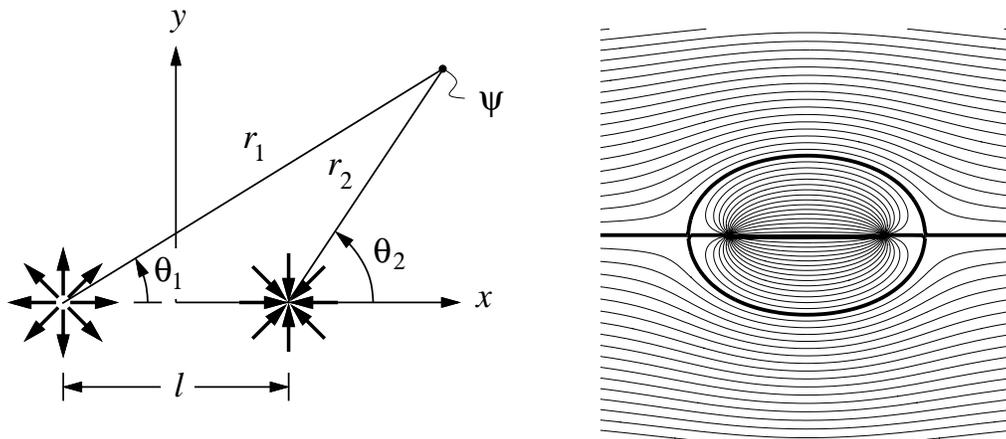


The bullet-shaped heavy line on the combined flow corresponds to the *dividing streamline*, which separates the fluid coming from the freestream and the fluid coming from the source. If we replace the dividing streamline by a solid semi-infinite body of the same shape, the flow about this body will be the same as the flow outside the dividing streamline in the superimposed flow.

## Uniform Flow with Source and Sink

We now superimpose a uniform flow in the  $x$ -direction, with a source located at  $(-\ell/2, 0)$ , and a sink of equal and opposite strength located at  $(+\ell/2, 0)$ , plus a freestream.

$$\psi = \frac{\Lambda}{2\pi} (\theta_1 - \theta_2) + V_\infty r \sin \theta$$

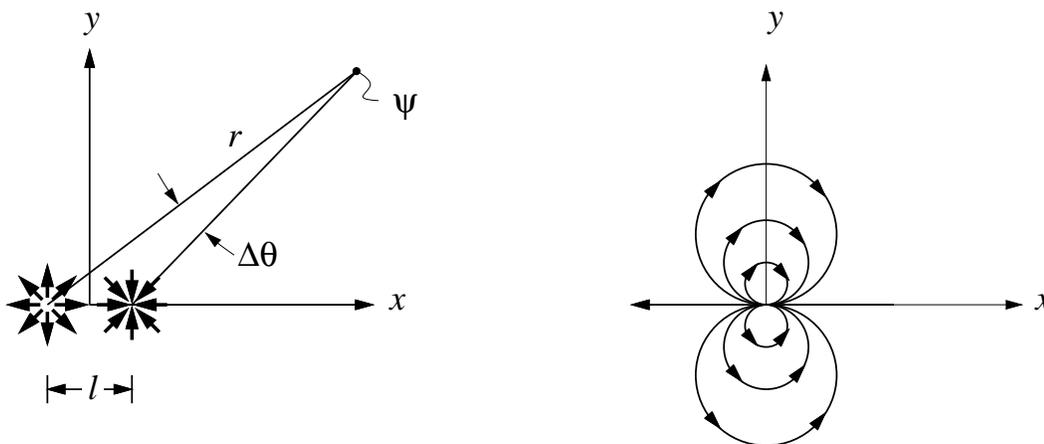


The figure on the right shows the streamlines of the combined flow. The heavy line again indicates the dividing streamline, which traces out a *Rankine oval*. All the streamlines inside the oval originate at the source on the left, and flow into the sink on the right. The net volume outflow from the oval is zero. Again, the dividing streamline could be replaced by a solid oval body of the same shape. The flow outside the oval then corresponds to the flow about this body.

## Doublet

Consider a source-sink pair with strengths  $\pm\Lambda$ , located at  $(\mp\ell/2, 0)$ . Now let the separation distance  $\ell$  approach zero, while simultaneously increasing the source and sink strengths such that the product  $\kappa \equiv \ell\Lambda$  remains constant. The resulting flow is a *doublet* with strength  $\kappa$ .

$$\psi = \lim_{\substack{\ell \rightarrow 0 \\ \kappa = \text{const.}}} -\frac{\kappa}{2\pi \ell} \Delta\theta = -\frac{\kappa}{2\pi} \frac{\sin \theta}{r}$$



A similar limiting process can be used to produce the doublet's potential function.

$$\phi = \frac{\kappa}{2\pi} \frac{\cos \theta}{r}$$

The streamline shapes of the doublet are obtained by setting

$$\begin{aligned} \psi &= -\frac{\kappa}{2\pi} \frac{\sin \theta}{r} = c = \text{constant} \\ r &= d \sin \theta \\ \text{where } d &= -\frac{\kappa}{2\pi c} \end{aligned}$$

In polar coordinates this is the equation for circles of diameter  $d$ , centered on  $x, y = (0, \pm d/2)$ .

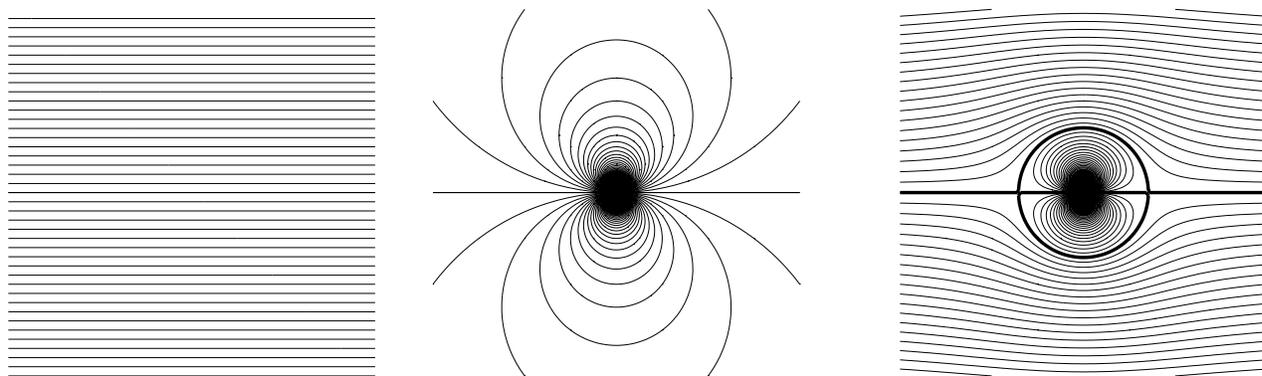
## Nonlifting Flow over Circular Cylinder

### Flowfield definition

We now superimpose a uniform flow with a doublet.

$$\begin{aligned} \psi &= V_\infty r \sin \theta - \frac{\kappa}{2\pi} \frac{\sin \theta}{r} = V_\infty r \sin \theta \left( 1 - \frac{\kappa}{2\pi V_\infty r^2} \right) \\ \text{or } \psi &= V_\infty r \sin \theta \left( 1 - \frac{R^2}{r^2} \right) \\ \text{where } R^2 &\equiv \kappa / (2\pi V_\infty) \end{aligned}$$

This corresponds to the flow about a circular cylinder of radius  $R$ .



The radial and tangential velocities can be obtained by differentiating the stream function as follows.

$$\begin{aligned} V_r &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty \cos \theta \left( 1 - \frac{R^2}{r^2} \right) \\ V_\theta &= -\frac{\partial \psi}{\partial r} = -V_\infty \sin \theta \left( 1 + \frac{R^2}{r^2} \right) \end{aligned}$$

### Surface velocities and pressures

On the surface of the cylinder where  $r = R$ , we have

$$\begin{aligned}V_r &= 0 \\V_\theta &= -2V_\infty \sin \theta\end{aligned}$$

The maximum surface speed of  $2V_\infty$  occurs at  $\theta = \pm 90^\circ$ .

The surface pressure is then obtained using the Bernoulli equation

$$p(\theta) = p_o - \frac{1}{2}\rho(V_r^2 + V_\theta^2)$$

Substituting  $V_r = 0$  and  $V_\theta(\theta)$ , and using the freestream value for the total pressure,

$$p_o = p_\infty + \frac{1}{2}\rho V_\infty^2$$

gives the following surface pressure distribution.

$$p(\theta) = p_\infty - \frac{1}{2}\rho V_\infty^2 (1 - 4 \sin^2 \theta)$$

The corresponding pressure coefficient is also readily obtained.

$$C_p(\theta) \equiv \frac{p(\theta) - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - 4 \sin^2 \theta$$