Fluids – Lecture 13 Notes

1. Bernoulli Equation
2. Uses of Bernoulli Equation

Reading: Anderson 3.2, 3.3

Bernoulli Equation

Derivation – 1-D case

The 1-D momentum equation, which is Newton's Second Law applied to fluid flow, is written as follows.

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}}$$

We now make the following assumptions about the flow.

- Steady flow: $\partial/\partial t = 0$
- Negligible gravity: $\rho g_x \simeq 0$
- Negligible viscous forces: $(F_x)_{\text{viscous}} \simeq 0$
- Low-speed flow: ρ is constant

These reduce the momentum equation to the following simpler form, which can be immediately integrated.

$$\begin{aligned} \rho u \frac{du}{dx} + \frac{dp}{dx} &= 0 \\ \frac{1}{2}\rho \frac{d(u^2)}{dx} + \frac{dp}{dx} &= 0 \\ \frac{1}{2}\rho u^2 + p &= \text{constant} \equiv p_o \end{aligned}$$

The final result is the one-dimensional *Bernoulli Equation*, which uniquely relates velocity and pressure if the simplifying assumptions listed above are valid. The constant of integration p_o is called the *stagnation pressure*, or equivalently the *total pressure*, and is typically set by known upstream conditions.

Derivation – 2-D case

The 2-D momentum equations are

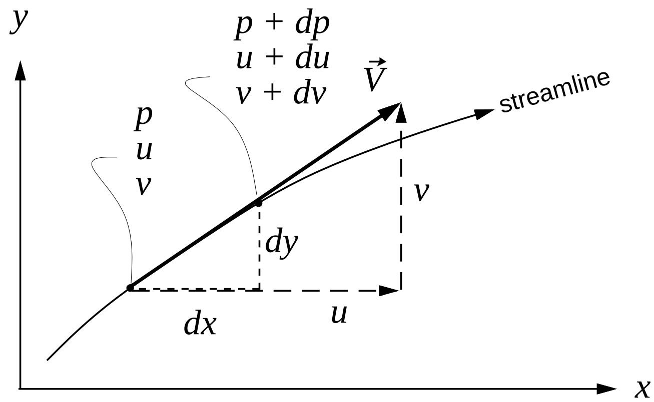
$$\begin{aligned} \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}} \\ \rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \rho g_y + (F_y)_{\text{viscous}} \end{aligned}$$

Making the same assumptions as before, these simplify to the following.

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0 \qquad (1)$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0 \qquad (2)$$

Before these can be integrated, we must first restrict ourselves only to flowfield variations along a streamline. Consider an incremental distance ds along the streamline, with projections dx and dy in the two axis directions. The speed V likewise has projections u and v.



Along the streamline, we have

$$\frac{dy}{dx} = \frac{v}{u}$$

or

$$u\,dy = v\,dx \tag{3}$$

We multiply the x-momentum equation (1) by dx, use relation (3) to replace $v\,dx$ by $u\,dy$, and combine the u-derivative terms into a du differential.

$$\begin{aligned}
\rho u \frac{\partial u}{\partial x}\,dx + \rho v \frac{\partial u}{\partial y}\,dx + \frac{\partial p}{\partial x}\,dx &= 0 \\
\rho u \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \right) + \frac{\partial p}{\partial x}\,dx &= 0 \\
\rho u\; du + \frac{\partial p}{\partial x}\,dx &= 0 \\
\frac{1}{2}\rho\, d\!\left(u^2\right) + \frac{\partial p}{\partial x}\,dx &= 0
\end{aligned} \tag{4}$$

We multiply the y-momentum equation (2) by dy, and performing a similar manipulation, we get

$$\begin{aligned}
\rho u \frac{\partial v}{\partial x}\,dy + \rho v \frac{\partial v}{\partial y}\,dy + \frac{\partial p}{\partial y}\,dy &= 0 \\
\rho v \left(\frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy \right) + \frac{\partial p}{\partial y}\,dy &= 0 \\
\rho v\; dv + \frac{\partial p}{\partial y}\,dy &= 0 \\
\frac{1}{2}\rho\, d\left(v^2\right) + \frac{\partial p}{\partial y}\,dy &= 0
\end{aligned} \tag{5}$$

Finally, we add equations (4) and (5), giving

$$\begin{aligned}
\frac{1}{2}\rho\, d\!\left(u^2 + v^2\right) + \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy &= 0 \\
\frac{1}{2}\rho\, d\!\left(u^2 + v^2\right) + dp &= 0
\end{aligned}$$

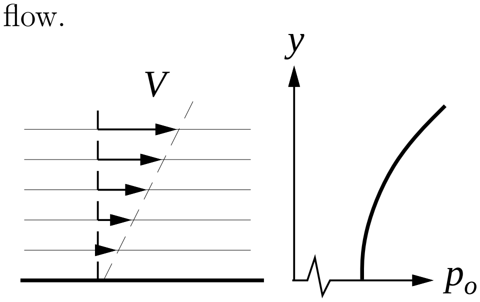
which integrates into the general Bernoulli equation

$$\frac{1}{2}\rho V^2 \; + \; p \;\; = \;\; \text{constant} \; \equiv \; p_o \qquad\qquad \text{(along a streamline)} \qquad (6)$$

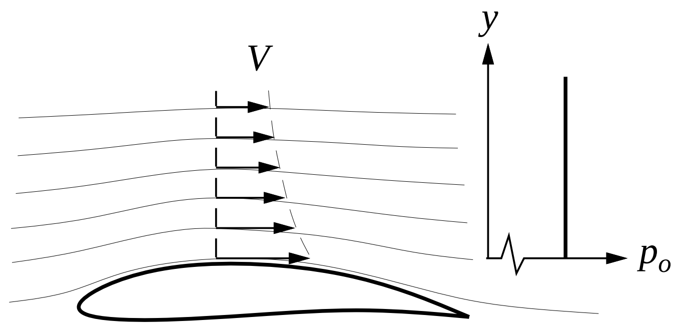
where $V^2 = u^2 + v^2$ is the square of the speed. For the 3-D case the final result is exactly the same as equation (6), but now the w velocity component is nonzero, and hence $V^2 = u^2 + v^2 + w^2$.

Irrotational Flow

Because of the assumptions used in the derivations above, in particular the streamline relation (3), the Bernoulli Equation (6) relates p and V only along any given streamline. Different streamlines will in general have different p_o constants, so p and V cannot be directly related between streamlines. For example, the simple shear flow on the left of the figure has parallel flow with a linear $u(y)$, and a uniform pressure p. Its p_o distribution is therefore parabolic as shown. Hence, there is no unique correspondence between velocity and pressure in such a flow.



Rotational flow



Irrotational flow

However, if the flow is irrotational, i.e. if $\vec{V} = \nabla\phi$ and $V^2 = |\nabla\phi|^2$, then p_o takes on the same value for all streamlines, and the Bernoulli Equation (6) becomes usable to relate p and V in the entire irrotational flowfield. Fortunately, a flowfield is irrotational if the upstream flow is irrotational (e.g. uniform), which is a very common occurance in aerodynamics. From the uniform far upstream flow we can evaluate

$$p_o \; = \; p_\infty + \frac{1}{2}\rho V_\infty^2 \; \equiv \; p_{o_\infty}$$

and the Bernoulli equation (6) then takes the more general form.

$$\frac{1}{2}\rho V^2 \; + \; p \;\; = \;\; p_{o_\infty} \qquad\qquad \text{(everywhere in an irrotational flow)} \qquad (7)$$

Uses of Bernoulli Equation

Solving potential flows

Having the Bernoulli Equantion (7) in hand allows us to devise a relatively simple two-step solution strategy for potential flows.

1. Determine the potential field $\phi(x, y, z)$ and resulting velocity field $\vec{V} = \nabla\phi$ using the

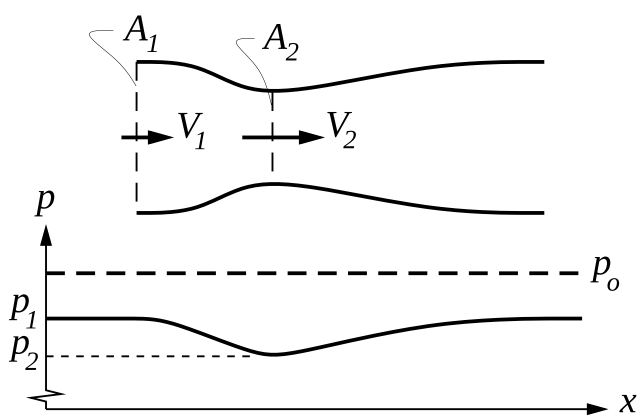
governing equations.

2. Once the velocity field is known, insert it into the Bernoulli Equation to compute the pressure field $p(x, y, z)$.

This two-step process is simple enough to permit very economical aerodynamic solution methods which give a great deal of physical insight into aerodynamic behavior. The alternative approaches which do not rely on Bernoulli Equation must solve for $\vec{V}(x, y, z)$ and $p(x, y, z)$ simultaneously, which is a tremendously more difficult problem which can be approached only through brute force numerical computation.

Venturi flow

Another common application of the Bernoulli Equation is in a *venturi*, which is a flow tube with a minimum cross-sectional area somewhere in the middle.



Assuming incompressible flow, with ρ constant, the mass conservation equation gives

$$A_1 V_1 \;=\; A_2 V_2 \tag{8}$$

This relates V_1 and V_2 in terms of the geometric cross-sectional areas.

$$V_2 \;=\; V_1 \frac{A_1}{A_2}$$

Knowing the velocity relationship, the Bernoulli Equation then gives the pressure relationship.

$$p_1 + \frac{1}{2}\rho V_1^2 \;=\; p_o \;=\; p_2 + \frac{1}{2}\rho V_2^2 \tag{9}$$

Equations (8) and (9) together can be used to determine the inlet velocity V_1, knowing only the pressure difference $p_1 - p_2$ and the geometric areas. By direct substution we have

$$V_1 \;=\; \sqrt{\frac{2(p_1 - p_2)}{\rho\left[(A_1/A_2)^2 - 1\right]}}$$

A venturi can therefore by used as an *airspeed indicator*, if some means of measuring the pressure difference $p_1 - p_2$ is provided.