

Fluids – Lecture 7 Notes

1. Momentum Flow
2. Momentum Conservation

Reading: Anderson 2.5

Momentum Flow

Before we can apply the principle of momentum conservation to a fixed permeable control volume, we must first examine the effect of flow through its surface. When material flows through the surface, it carries not only mass, but momentum as well. The *momentum flow* can be described as

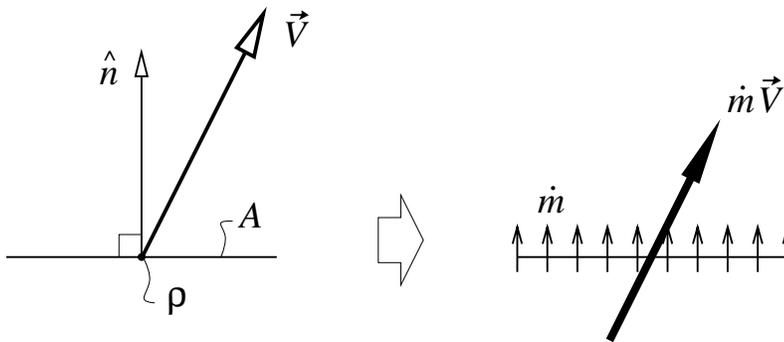
$$\overrightarrow{\text{momentum flow}} = (\overrightarrow{\text{mass flow}}) \times (\overrightarrow{\text{momentum / mass}})$$

where the mass flow was defined earlier, and the momentum/mass is simply the velocity vector \vec{V} . Therefore

$$\overrightarrow{\text{momentum flow}} = \dot{m} \vec{V} = \rho (\vec{V} \cdot \hat{n}) A \vec{V} = \rho V_n A \vec{V}$$

where $V_n = \vec{V} \cdot \hat{n}$ as before. Note that while mass flow is a scalar, the momentum flow is a vector, and points in the same direction as \vec{V} . The momentum flux vector is defined simply as the momentum flow per area.

$$\overrightarrow{\text{momentum flux}} = \rho V_n \vec{V}$$

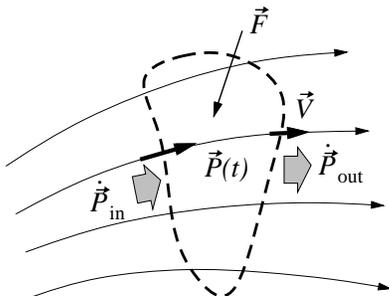


Momentum Conservation

Newton's second law states that during a short time interval dt , the impulse of a force \vec{F} applied to some affected mass, will produce a momentum change $d\vec{P}_a$ in that affected mass. When applied to a fixed control volume, this principle becomes

$$\frac{d\vec{P}_a}{dt} = \vec{F} \quad (1)$$

$$\frac{d\vec{P}}{dt} + \dot{\vec{P}}_{\text{out}} - \dot{\vec{P}}_{\text{in}} = \vec{F} \quad (2)$$



In the second equation (2), \vec{P} is defined as the instantaneous momentum inside the control volume.

$$\vec{P}(t) \equiv \iiint \rho \vec{V} dV$$

The $\dot{\vec{P}}_{\text{out}}$ is added because mass leaving the control volume carries away momentum provided by \vec{F} , which \vec{P} alone doesn't account for. The $\dot{\vec{P}}_{\text{in}}$ is subtracted because mass flowing into the control volume is incorrectly accounted in \vec{P} , and hence must be discounted. Both terms are evaluated by a surface integral of the momentum flux over the entire boundary.

$$\dot{\vec{P}}_{\text{out}} - \dot{\vec{P}}_{\text{in}} = \oiint \rho (\vec{V} \cdot \hat{n}) \vec{V} dA$$

The sign of $\vec{V} \cdot \hat{n}$ automatically accounts for both inflow and outflow.

Applied forces

The force \vec{F} consists of two types.

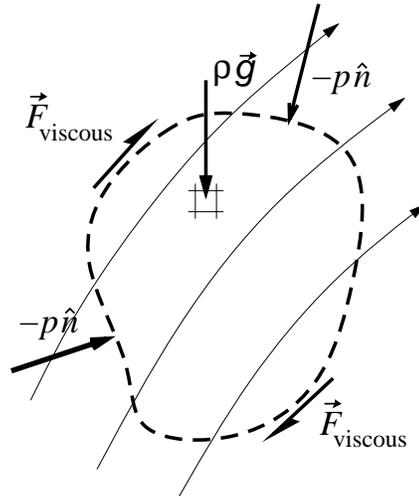
Body forces. These act on fluid inside the volume. The most common example is the gravity force, along the gravitational acceleration vector \vec{g} .

$$\vec{F}_{\text{gravity}} = \iiint \rho \vec{g} dV$$

Surface forces. These act on the surface of the volume, and can be separated into pressure and viscous forces.

$$\vec{F}_{\text{pressure}} = \oiint -p \hat{n} dA$$

The viscous force is complicated to write out, and for now will simply be called \vec{F}_{viscous} .



Integral Momentum Equation

Substituting all the momentum, momentum flow, and force definitions into Newton's second law (2) gives the *Integral Momentum Equation*.

$$\frac{d}{dt} \iiint \rho \vec{V} dV + \oiint \rho (\vec{V} \cdot \hat{n}) \vec{V} dA = \oiint -p \hat{n} dA + \iiint \rho \vec{g} dV + \vec{F}_{\text{viscous}} \quad (3)$$

Along with the Integral Mass Equation, this equation can be applied to solve many problems involving finite control volumes.

Differential Momentum Equation

The pressure surface integral in equation (3) can be converted to a volume integral using the Gradient Theorem.

$$\oint p \hat{n} dA = \iiint \nabla p d\mathcal{V}$$

The momentum-flow surface integral is also similarly converted using Gauss's Theorem. This integral is a vector quantity, and for clarity the conversion is best done on each component separately. After substituting $\vec{V} = u \hat{i} + v \hat{j} + w \hat{k}$, we have

$$\begin{aligned} \oint \rho (\vec{V} \cdot \hat{n}) (u \hat{i} + v \hat{j} + w \hat{k}) dA &= \hat{i} \iiint \nabla \cdot (\rho \vec{V} u) d\mathcal{V} \\ &+ \hat{j} \iiint \nabla \cdot (\rho \vec{V} v) d\mathcal{V} \\ &+ \hat{k} \iiint \nabla \cdot (\rho \vec{V} w) d\mathcal{V} \end{aligned}$$

The x -component of the integral momentum equation (3) can now be written strictly in terms of volume integrals.

$$\iiint \left[\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) + \frac{\partial p}{\partial x} - \rho g_x - (F_x)_{\text{viscous}} \right] d\mathcal{V} = 0 \quad (4)$$

This relation must hold for *any* control volume whatsoever. If we place an infinitesimal control volume at every point in the flow and apply equation (4), we can see that the whole quantity in the brackets must be zero at every point. This results in the *x-Momentum Equation*

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}} \quad (5)$$

and the y - and z -Momentum Equations follow by the same process.

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \vec{V}) = -\frac{\partial p}{\partial y} + \rho g_y + (F_y)_{\text{viscous}} \quad (6)$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{V}) = -\frac{\partial p}{\partial z} + \rho g_z + (F_z)_{\text{viscous}} \quad (7)$$

These three equations are the embodiment of the Newton's second law of motion, applied at every point in the flowfield. The steady flow version has the $\partial/\partial t$ terms omitted.