NAME Solution

Unified Quiz S6

April 22, 2004

One 8 1/2" x 11" sheet (two sides) of notes
Calculators allowed.
Calculators may be used for arithmetic only.
No books allowed.

- Put your name on each page of the exam.
- Read all questions carefully.
- Do all work for each problem on the two pages provided.
- Show intermediate results.
- Explain your work --- don't just write equations. Any problem without an explanation can receive no better than a "B" grade.
- Partial credit will be given, but only when the intermediate results and explanations are clear.
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Show appropriate units with your final answers.
- Box your final answers.

Exam Scoring

#1 (25%)	
#2 (25%)	
#3 (25%)	
#4 (25%)	
Total	

Problem 1 Name SOLUTION

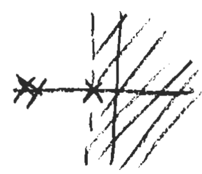
A causal, LTI system, G, has impulse response $g(t)$. The Laplace transform of $g(t)$ is

$$G(s) = \frac{4}{(s+1)^2(s+3)}$$

1. What is the region of convergence of the Laplace transform? Explain.
2. Is the system stable or unstable? Explain.
3. Find $g(t)$.

1. The system has poles @ $s = -1$ and $s = -3$. Because the system is causal, the R.O.C. must be to the right of the rightmost pole. So the R.O.C. is

$$\mathrm{Re}[s] > -1$$



2. The R.O.C. contains $\mathrm{Re}[s] = 0$. Therefore, it is stable

3. Do a partial fraction expansion:

$$G(s) = \frac{4}{(s+1)^2(s+3)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{c}{s+3}$$

Use coverup method to find b, c:

$$b = \left.\frac{4}{s+3}\right|_{s=-1} = \frac{4}{2} = 2$$

$$c = \left.\frac{4}{(s+1)^2}\right|_{s=-3} = \frac{4}{(-2)^2} = 1$$

Problem 1 Name SOLUTION

So

$$\frac{4}{(s+1)^2(s+3)} = \frac{a}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{s+3}$$

This is true for all s, so true at $s=0$:

$$\frac{4}{1^2 \cdot 3} = \frac{a}{1} + \frac{2}{1^2} + \frac{1}{3}$$

$$\Rightarrow \quad a = \frac{4}{3} - 2 - \frac{1}{3} = -1$$

$$\Rightarrow \quad G(s) = \frac{-1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{s+3}, \qquad \mathcal{R}[s] > -1$$

So the inverse LT is

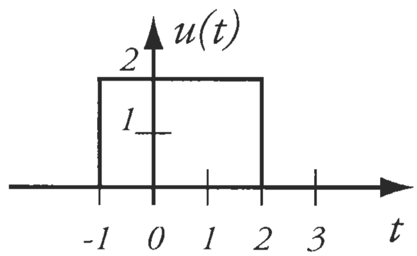
$$\boxed{g(t) = \left[-e^{-t} + 2te^{-t} + e^{-3t} \right] \sigma(t)}$$

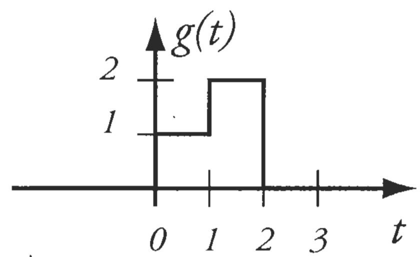
Problem 2 Name SOLUTION

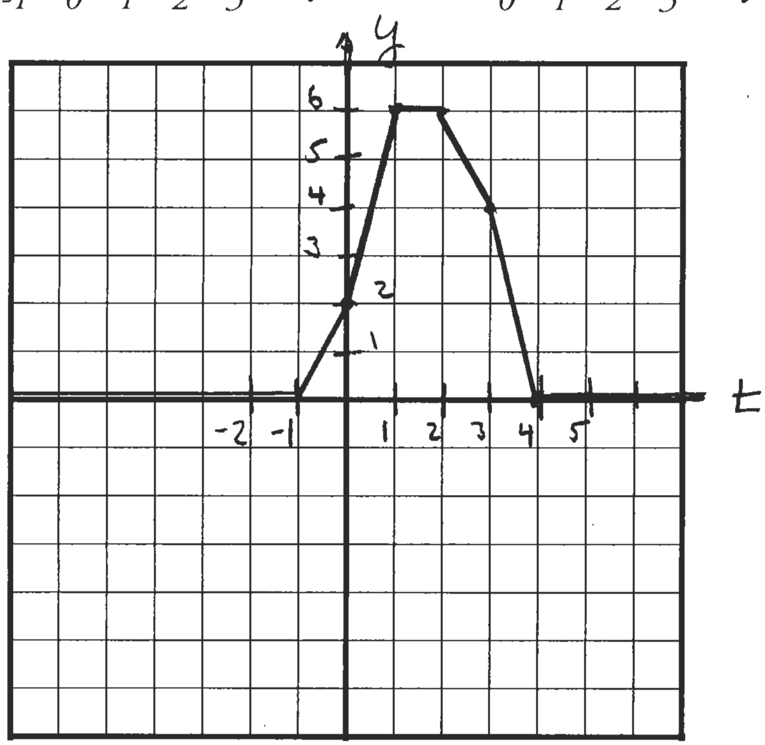
Given the signals $g(t)$ and $u(t)$ as plotted below, find the signal $y(t)$ given by

$$y(t) = g(t) * u(t)$$

Sketch the result in the grid below, as accurately as possible. Be sure to label the axes of the grid. Explain your reasoning on the page that follows.

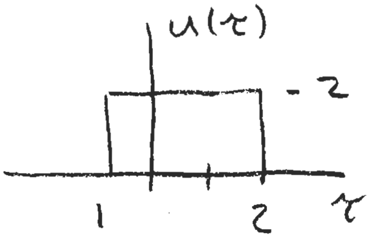


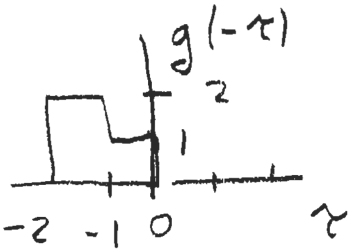




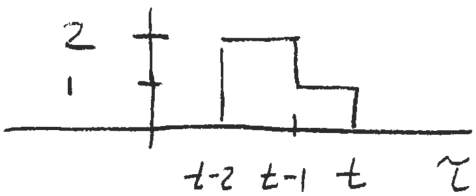
Problem 2 Name SOLUTION

Use flip & slide to do convolution





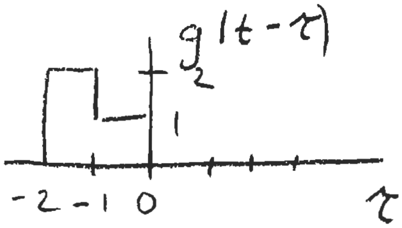
So $g(t-\tau)$ is

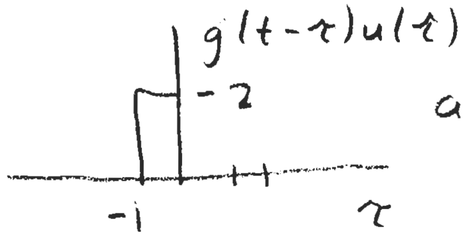


For $t \geq 4$ or $t \leq -1$, there is no overlap of $u(\tau)$ & $g(t-\tau)$, so $y(t)=0$

Do several values of t:

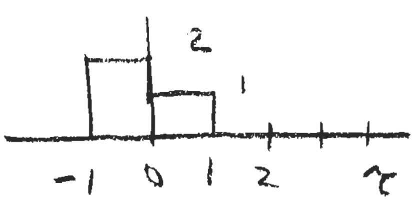
<u>$t = 0$</u>

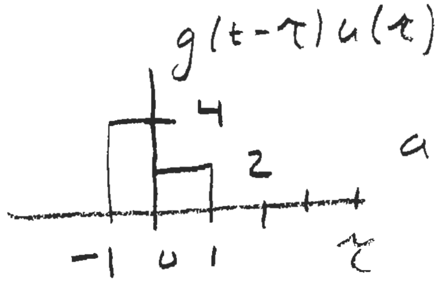




area = $y(t) = 2$

<u>$t = 1$</u>





area = $y(t) = 6$

Problem 2 Name SOLUTION

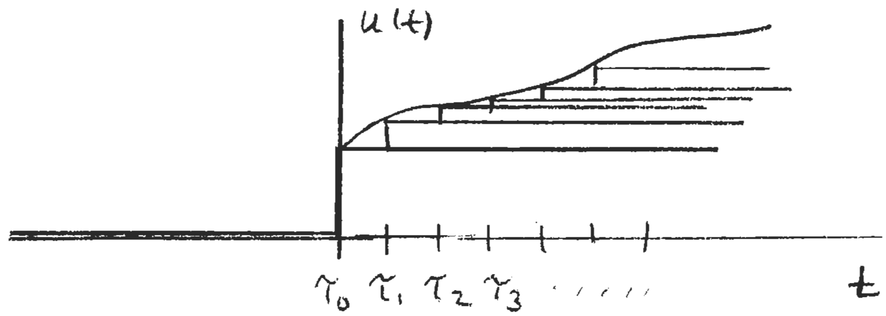
Similarly, $y(2) = 6$, $y(3) = 4$.

Since u, g are piecewise constant, y is piecewise linear and continuous. So $y(t)$ is as shown in graph.

Problem 3 Name SOLUTION

Consider an LTI system G with input signal $u(t)$ and output signal $y(t)$. Explain why knowing the step response of the system allows one to determine the response of the system to an arbitrary input $u(t)$. You should do more than just give the equation for $y(t)$ — you should explain why the result is true.

An arbitrary signal $u(t)$ can be approximated arbitrarily well as a sum of delayed and scaled steps, as shown in the figure:



(The $u(t)$ shown has a discontinuity a $t=\tau_0=0$. This is not necessary for the argument)

$U(t)$ is approximately

$$u(t) \approx u(0)\,\sigma(t) + \sum_{n=1}^{\infty}\left[u(\tau_n)-u(\tau_{n-1})\right]\sigma(t-\tau_n)$$

The response $y(t)$ can be found by superposition, since the system is linear and time invariant, and we know the step response:

$$y(t) \approx u(0)\,g_s(t) + \sum_{n=1}^{\infty}\left[u(\tau_n)-u(\tau_{n-1})\right]g_s(t-\tau_n)$$

Problem 3 Name SOLUTION

$$= u(0)\, g_s(t) + \sum_{n=1}^{\infty} \frac{[u(\tau_n) - u(\tau_{n-1})]}{\tau_n - \tau_{n-1}} g_s(t-\tau_n)[\tau_n - \tau_{n-1}]$$

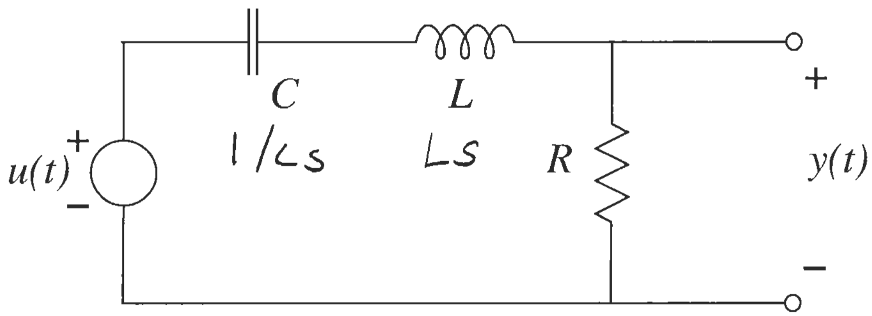
In the limit as $\tau_n - \tau_{n-1} \to 0$, the sum becomes the integral, and the ratio becomes a derivative, so

$$y(t) = g_s(t)\, u(0) + \int_0^{\infty} \frac{du(\tau)}{d\tau} g_s(t-\tau)\, d\tau$$

<u>Duhamel's</u> <u>integral</u> expresses the response to an arbitrary input in terms of the step response.

Problem 4 Name SOLUTION

Find the step response of the circuit below. The component values are $C = 0.5$ F, $L = 1$ H, and $R = 3\ \Omega$.



To find the transfer function, assume $u(t) = U e^{st}$, $y(t) = Y e^{st}$, and the components have impedances as shown. Then the transfer function is

$$\frac{Y}{U} = G(s) = \frac{R}{R + Ls + 1/Cs},$$

since the circuit is a voltage divider.

Simplifying,

$$G(s) = \frac{RCs}{LCs^2 + RCs + 1}$$

$$= \frac{(R/L)\, s}{s^2 + \frac{R}{L} s + \frac{1}{LC}} = \frac{3s}{s^2 + 3s + 2}$$

Problem 4 Name SOLUTION

If the input is a unit step, $u(t) = \sigma(t)$, then $U(s) = 1/s$ $(s > 0)$. Therefore,

$$Y(s) = G(s)\,U(s) = \frac{G(s)}{s} = \frac{3}{s^2 + 3s + 2}$$

$$= \frac{3}{(s+1)(s+2)} \quad \text{(factoring)}$$

$$= \frac{3}{s+1} - \frac{3}{s+2} \quad \text{(partial fractions)}$$

Since the system is causal, this implies

$$\boxed{g_s(t) = y(t) = \left[3e^{-t} - 3e^{-2t}\right]\sigma(t)}$$