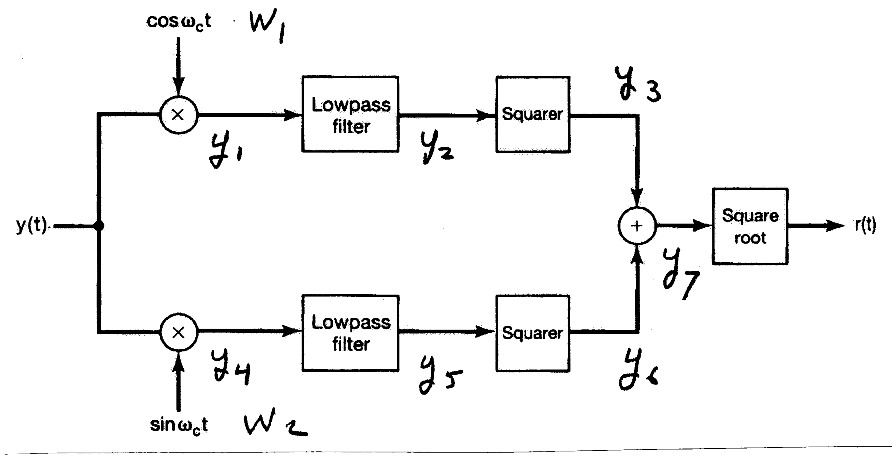
Unified Engineering II **Spring 2004**

Problem S17 Solution

To begin, label the signals as shown below:



From the problem statement,

$$y(t) = [x(t) + A]\cos(2\pi f_c t + \theta_c)$$

Define

$$\begin{aligned} z(t) &= x(t) + A \\ w(t) &= \cos(2\pi f_c t + \theta_c) \end{aligned}$$

The factor $w(t)$ can be expanded as

$$w(t) = \cos(2\pi f_c t + \theta_c) = \cos\theta_c \, \cos 2\pi f_c t - \sin\theta_c \, \sin 2\pi f_c t$$

The Fourier transform of $w(t)$ is then given by

$$\begin{aligned} W(f) &= \mathcal{F}[\cos(2\pi f_c t + \theta_c)] \\ &= \frac{1}{2}\cos\theta_c \, [\delta(f - f_c) + \delta(f + f_c)] - \frac{1}{2}\sin\theta_c \, [-j\delta(f - f_c) + j\delta(f + f_c)] \\ &= \frac{1}{2}(\cos\theta_c + j\sin\theta_c)\,\delta(f - f_c) + \frac{1}{2}(\cos\theta_c - j\sin\theta_c)\,\delta(f + f_c) \end{aligned}$$

The Fourier transform of $z(t) = x(t) + A$ is given by

$$Z(f) = \mathcal{F}[z(t)] = X(f) + A\delta(f)$$

$Z(f)$ is bandlimited, because $X(f)$ is, and of course the impulse function is bandlimited. So the FT of $y(t)$ is given by the convolution

$$\begin{aligned} Y(w) &= Z(f) * W(f) \\ &= \frac{1}{2}\left[(\cos\theta_c + j\sin\theta_c)\,Z(f - f_c) + (\cos\theta_c - j\sin\theta_c)\,Z(f + f_c)\right] \end{aligned}$$

Next, compute the spectra of $y_1(t)$ and $y_2(t)$. To do so, we need the spectra of $w_1(t)$ and $w_2(t)$:

$$\begin{aligned}
W_1(f) = \mathcal{F}[w_1(t)] &= \mathcal{F}[\cos 2\pi f_c t] \\
&= \frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)] \\
W_2(f) = \mathcal{F}[w_2(t)] &= \mathcal{F}[\sin 2\pi f_c t] \\
&= \frac{1}{2}[-j\delta(f-f_c)+j\delta(f+f_c)]
\end{aligned}$$

Then

$$\begin{aligned}
Y_1(f) &= W_1(f) * Y(f) \\
&= \frac{1}{2}[Y(f-f_c)+Y(f-f_c)] \\
&= \frac{1}{4}[(\cos\theta_c + j\sin\theta_c)\,Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c)\,Z(f)] \\
&\quad + \frac{1}{4}[(\cos\theta_c + j\sin\theta_c)\,Z(f) + (\cos\theta_c - j\sin\theta_c)\,Z(f+2f_c)] \\
&= \frac{1}{2}\cos\theta_c\,Z(f) \\
&\quad + \frac{1}{4}[(\cos\theta_c + j\sin\theta_c)\,Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c)\,Z(f+2f_c)]
\end{aligned}$$

Similarly,

$$\begin{aligned}
Y_4(f) &= W_2(f) * Y(f) \\
&= \frac{1}{2}[-jY(f-f_c)+jY(f-f_c)] \\
&= \frac{-j}{4}[(\cos\theta_c + j\sin\theta_c)\,Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c)\,Z(f)] \\
&\quad + \frac{j}{4}[(\cos\theta_c + j\sin\theta_c)\,Z(f) + (\cos\theta_c - j\sin\theta_c)\,Z(f+2f_c)] \\
&= -\frac{1}{2}\sin\theta_c\,Z(f) \\
&\quad + \frac{1}{4}[(-j\cos\theta_c + \sin\theta_c)\,Z(f-2f_c) + (j\cos\theta_c + \sin\theta_c)\,Z(f+2f_c)]
\end{aligned}$$

Now, when $y_1(t)$ and $y_4(t)$ are passed through the lowpass filters, the $Z(f-2f_c)$ and $Z(f+2f_c)$ terms are eliminated, and the $Z(f)$ terms are passed. Therefore,

$$\begin{aligned}
Y_2(f) &= \frac{1}{2}\cos\theta_c\,Z(f) \\
Y_5(f) &= -\frac{1}{2}\sin\theta_c\,Z(f)
\end{aligned}$$

and

$$
\begin{aligned}
y_2(t) &= \frac{1}{2}\cos\theta_c\, z(t) \\
y_5(t) &= -\frac{1}{2}\sin\theta_c\, z(t)
\end{aligned}
$$

After passing these signals through the squarers, we have

$$
\begin{aligned}
y_3(t) &= \frac{1}{4}\cos^2\theta_c\, z^2(t) \\
y_6(t) &= \frac{1}{4}\sin^2\theta_c\, z^2(t)
\end{aligned}
$$

$y_7(t)$ is the sum of these, so that

$$
\begin{aligned}
y_7(t) &= y_3(t) + y_7(t) \\
&= \frac{1}{4}\left[\cos^2\theta_c\, z^2(t) + \sin^2\theta_c\, z^2(t)\right] \\
&= \frac{1}{4} z^2(t)
\end{aligned}
$$

Finally, $r(t)$ is obtained by passing taking the square root of $y_7(t)$, so that

$$
\begin{aligned}
r(t) &= \sqrt{z^2(t)/4} \\
&= \frac{|z(t)|}{2}
\end{aligned}
$$

if the positive root is always taken. But $z(t) = x(t) + A$ is always positive, according to the problem statement. Therefore,

$$x(t) = 2r(t) - A$$