Unified Engineering II **Spring 2004**

Problem S16 Solution

The Fourier transform of $x(t)$ is given by $X(f)$. Then the FT of $x_1(t)$ is given by

$$X_1(f) = H(f)X(f) = \begin{cases} -jX(f), & 0 < f < f_M \\ +jX(f), & -f_M < f < 0 \\ 0, & |f| > f_M \end{cases}$$

The signal $x_2(t)$ is given by

$$x_2(t) = w_1(t)x_1(t)$$

where $w_1(t) = \cos 2\pi f_c t$. The FT of $w_1(t)$ is

$$W_1(f) = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

The FT of $x_2(t)$ is then

$$\begin{aligned}
X_2(f) &= X_1(f) * W_1(f) \\
&= \frac{1}{2}[X_1(f-f_c) + X_1(f+f_c)] \\
&= \begin{cases} -\frac{j}{2}X(f-f_c), & f_c < f < f_c + f_M \\ +\frac{j}{2}X(f-f_c), & f_c - f_M < f < f_c \\ -\frac{j}{2}X(f+f_c), & -f_c < f < -f_c + f_M \\ +\frac{j}{2}X(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases}
\end{aligned}$$

The signal $x_3(t)$ is given by

$$x_3(t) = w_2(t)x(t)$$

where $w_2(t) = \sin 2\pi f_c t$. The FT of $w_2(t)$ is

$$W_2(f) = \frac{1}{2}[-j\delta(f-f_c) + j\delta(f+f_c)]$$

The FT of $x_3(t)$ is then

$$\begin{aligned}
X_3(f) &= X(f) * W_2(f) \\
&= \frac{1}{2}[-jX(f-f_c) + jX(f+f_c)] \\
&= \begin{cases} -\frac{j}{2}X(f-f_c), & f_c < f < f_c + f_M \\ -\frac{j}{2}X(f-f_c), & f_c - f_M < f < f_c \\ +\frac{j}{2}X(f+f_c), & -f_c < f < -f_c + f_M \\ +\frac{j}{2}X(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases}
\end{aligned}$$

Finally, the FT of $y(t)$ is given by

$$\begin{aligned}
Y(f) &= X_2(f) + X_3(f) \\
&= \begin{cases} -jX(f-f_c), & f_c < f < f_c + f_M \\ 0, & f_c - f_M < f < f_c \\ 0, & -f_c < f < -f_c + f_M \\ +jX(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} -jX(f-f_c), & f_c < f < f_c + f_M \\ +jX(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases}
\end{aligned}$$

First, $y(t)$ is guaranteed to be real if $x(t)$, because if $x(t)$ real, $X(f)$ has conjugate symmetry, and then $Y(f)$ has conjugate symmetry, which implies $y(t)$ real.

Second, $x(t)$ can be recovered from $y(t)$s as follows. If $y(t)$ is modulated by $2\sin 2\pi f_c t$, the resulting signal is $z(t) = 2y(t)\sin 2\pi f_c t$, which has FT

$$\begin{aligned} Z(f) &= -jY(f-f_c) + jY(f+f_c) \\ &= \begin{cases} -X(f-2f_c), & 2f_c < f < 2f_c + f_M \\ +X(f), & -f_M < f < 0 \\ +X(f), & 0 < f < f_M \\ -X(f+2f_c), & -2f_c - f_M < f < -2f_c \\ 0, & \text{else} \end{cases} \end{aligned}$$

If $z(t)$ is then passed through a lowpass filter, with cutoff at $f = \pm f_M$, then the resulting signal is identical to $x(t)$.