

Problem S8 Solution

1. The convolution is given by

$$y(t) = g(t) * u(t) = \int_{-\infty}^{\infty} g(t - \tau)u(\tau) d\tau \quad (1)$$

Note that $u(\tau)$ is nonzero only for $-3 \leq \tau \leq 0$, and $g(t - \tau)$ is nonzero only for $0 \leq t - \tau \leq 3$, that is, for $-3 + t \leq \tau \leq t$. So there are four distinct regimes:

- (a) $t < -3$
- (b) $-3 \leq t \leq 0$
- (c) $0 \leq t \leq 3$
- (d) $t > 3$

For cases (a) and (d), there is no overlap between $g(t - \tau)$ and $u(\tau)$, so $y(t) = 0$. For case (b), the overlap is for $-3 \leq \tau \leq t$. So

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(t - \tau)u(\tau) d\tau \\ &= \int_{-3}^t \sin(-2\pi(t - \tau)) \sin(2\pi\tau) d\tau \end{aligned}$$

At this point, we have to do a little trig:

$$\begin{aligned} \sin(-2\pi(t - \tau)) \sin(2\pi\tau) &= \sin(2\pi(\tau - t)) \sin(2\pi\tau) \\ &= [\sin(2\pi\tau) \cos(2\pi t) - \cos(2\pi\tau) \sin(2\pi t)] \sin(2\pi\tau) \\ &= \cos(2\pi t) \sin^2(2\pi\tau) - \sin(2\pi t) \cos(2\pi\tau) \sin(2\pi\tau) \\ &= \cos(2\pi t) \frac{1 - \cos(4\pi\tau)}{2} - \sin(2\pi t) \frac{\sin(4\pi\tau)}{2} \end{aligned}$$

So the integral is given by

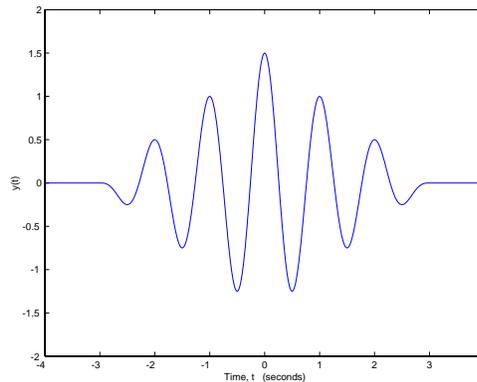
$$\begin{aligned} y(t) &= \int_{-3}^t \frac{\cos(2\pi t)}{2} d\tau - \int_{-3}^t \frac{\cos(2\pi t)}{2} \cos(4\pi\tau) d\tau - \int_{-3}^t \frac{\sin(2\pi t)}{2} \sin(4\pi\tau) d\tau \\ &= \frac{\cos(2\pi t)}{2} (t + 3) - \frac{\cos(2\pi t)}{8\pi} \sin(4\pi\tau) \Big|_{\tau=-3}^t + \frac{\sin(2\pi t)}{8\pi} \cos(4\pi\tau) \Big|_{\tau=-3}^t \\ &= \frac{\cos(2\pi t)}{2} (t + 3) - \frac{\cos(2\pi t)}{8\pi} \sin(4\pi t) + \frac{\sin(2\pi t)}{8\pi} [\cos(4\pi t) - 1] \end{aligned}$$

(As often happens with problems involving trig functions, there are other equivalent expressions.)

For case (c), the region of integration is $-3 + t \leq \tau \leq 0$. So

$$\begin{aligned}
 y(t) &= \int_{-3+t}^0 \frac{\cos(2\pi t)}{2} d\tau - \int_{-3+t}^0 \frac{\cos(2\pi t)}{2} \cos(4\pi\tau) d\tau - \int_{-3+t}^0 \frac{\sin(2\pi t)}{2} \sin(4\pi\tau) d\tau \\
 &= \frac{\cos(2\pi t)}{2}(3-t) - \frac{\cos(2\pi t)}{8\pi} \sin(4\pi\tau) \Big|_{\tau=-3+t}^0 + \frac{\sin(2\pi t)}{8\pi} \cos(4\pi\tau) \Big|_{\tau=-3+t}^0 \\
 &= \frac{\cos(2\pi t)}{2}(3-t) + \frac{\cos(2\pi t)}{8\pi} \sin(4\pi t) - \frac{\sin(2\pi t)}{8\pi} [\cos(4\pi t) - 1]
 \end{aligned}$$

2. $y(t)$ is plotted below.



3. The maximum value of $y(t - T)$ occurs at time T . So I would use this center peak to identify the delay time T .
4. The adjacent peaks are nearly as tall as the center peak, so if noise were added to the signal, the tallest peak might not be the center peak, so we might use the wrong peak to determine the delay time.
5. The chirp signal of Problem S6 produces an ambiguity function with only one prominent peak. Therefore, the addition of noise should not make it difficult to accurately determine the delay time.