1. The differential equation is

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6\, y(t) = u(t) = \sigma(t)$$

Find the homogeneous and particular solutions:

<u>homogeneous</u>:

Assume $y(t) = Y e^{st}$. Then

$$s^2 Y + 5s\, Y + 6Y = 0$$

$$\Rightarrow \quad s^2 + 5s + 6 = 0$$

$$\Rightarrow \quad (s+2)(s+3) = 0$$

$$\Rightarrow \quad s_1 = -2, \quad s_2 = -3$$

The homogeneous solution is therefore

$$y_h(t) = a e^{-2t} + b e^{-3t}$$

<u>particular</u>:

Since $u(t) = \sigma(t)$, $u(t) = 1 = \text{constant}$ for $t \geq 0$. Therefore, assume

$$y_p(t) = c = \text{constant}$$

Plugging into the differential equation,

$$6c = 1 \quad \Rightarrow \quad c = 1/6$$

<u>total solution</u>:

The total solution is

$$y(t) = y_p(t) + y_h(t) = 1/6 + a e^{-2t} + b e^{-3t}$$

The ICs are $y(0)=0$, $y'(0)=0$. Therefore,

$$a+b = -1/6$$
$$-2a-3b = 0$$

Solving,

$$a = -1/2$$
$$b = 1/3$$

Therefore,

$$\boxed{\begin{aligned} g_s(t) &= \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}, & t>0 \\ &= 0, & t<0 \end{aligned}}$$

2. $$g(t) = \frac{d}{dt} g_s(t)$$
$$\begin{aligned} &= e^{-2t} - e^{-3t} & t>0 \\ &= 0 & t<0 \\ &= 0 & t=0 \end{aligned}$$

The last part is because $g_s(t)$ has no discontinuity at $t=0$. Therefore,

$$\boxed{g(t) = \sigma(t)\left[e^{-2t} - e^{-3t}\right]}$$

3. Since the input is an exponential, it makes sense to guess

$$y(t) = c\,e^{-2t}$$

If we plug this into the D.E, we obtain

$$4ce^{-2t} - 10ce^{-2t} + 6ce^{-2t} = e^{-2t}$$

$$\Rightarrow \quad 0 = e^{-2t}$$

But this is not possible. So our guess doesn't work.

As we'll see below, a better guess is

$$y_p(t) = c\,t\,e^{-2t}$$

4. $$y(t) = \int_0^t g(t-\tau)\,u(\tau)\,d\tau$$

$$= \int_0^t \left[e^{-2(t-\tau)} - e^{-3(t-\tau)}\right] e^{-2\tau}\,d\tau$$

$$= \int_0^t \left[e^{-2t} - 3e^{-3t+\tau}\right] d\tau$$

$$= e^{-2t}\int_0^t dt - 3e^{-3t}\int_0^t e^{\tau}\,d\tau$$

$$= e^{-2t}\cdot t - 3e^{-3t}\cdot\left(e^{t}-1\right)$$

Therefore,

$$\boxed{y(t) = \left[3e^{-2t} - 3e^{-3t} + t\,e^{-2t}\right]\sigma(t)}$$

homogeneous particular

So the response to an exponential is not always an exponential — sometimes it includes a <u>secular</u> term (one with a factor of t)