Unified Engineering II **Spring 2004**

Problem S16 Solution

The Fourier transform of $x(t)$ is given by $X(f)$. Then the FT of $x_1(t)$ is given by

$$X_1(f) = H(f)X(f) = \begin{cases} -jX(f), & 0 < f < f_M \\ +jX(f), & -f_M < f < 0 \\ 0, & |f| > f_M \end{cases}$$

The signal $x_2(t)$ is given by

$$x_2(t) = w_1(t)x_1(t)$$

where $w_1(t) = \cos 2\pi f_c t$. The FT of $w_1(t)$ is

$$W_1(f) = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

The FT of $x_2(t)$ is then

$$\begin{aligned} X_2(f) &= X_1(f) * W_1(f) \\ &= \frac{1}{2}[X_1(f-f_c) + X_1(f+f_c)] \\ &= \begin{cases} -\frac{j}{2}X(f-f_c), & f_c < f < f_c + f_M \\ +\frac{j}{2}X(f-f_c), & f_c - f_M < f < f_c \\ -\frac{j}{2}X(f+f_c), & -f_c < f < -f_c + f_M \\ +\frac{j}{2}X(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases} \end{aligned}$$

The signal $x_3(t)$ is given by

$$x_3(t) = w_2(t)x(t)$$

where $w_2(t) = \sin 2\pi f_c t$. The FT of $w_2(t)$ is

$$W_2(f) = \frac{1}{2}[-j\delta(f-f_c) + j\delta(f+f_c)]$$

The FT of $x_3(t)$ is then

$$\begin{aligned} X_3(f) &= X(f) * W_2(f) \\ &= \frac{1}{2}[-jX(f-f_c) + jX(f+f_c)] \\ &= \begin{cases} -\frac{j}{2}X(f-f_c), & f_c < f < f_c + f_M \\ -\frac{j}{2}X(f-f_c), & f_c - f_M < f < f_c \\ +\frac{j}{2}X(f+f_c), & -f_c < f < -f_c + f_M \\ +\frac{j}{2}X(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases} \end{aligned}$$

Finally, the FT of $y(t)$ is given by

$$\begin{aligned} Y(f) &= X_2(f) + X_3(f) \\ &= \begin{cases} -jX(f-f_c), & f_c < f < f_c + f_M \\ 0, & f_c - f_M < f < f_c \\ 0, & -f_c < f < -f_c + f_M \\ +jX(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases} \\ &= \begin{cases} -jX(f-f_c), & f_c < f < f_c + f_M \\ +jX(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases} \end{aligned}$$

First, $y(t)$ is guaranteed to be real if $x(t)$, because if $x(t)$ real, $X(f)$ has conjugate symmetry, and then $Y(f)$ has conjugate symmetry, which implies $y(t)$ real.

Second, $x(t)$ can be recovered from $y(t)$s as follows. If $y(t)$ is modulated by $2\sin 2\pi f_c t$, the resulting signal is $z(t) = 2y(t)\sin 2\pi f_c t$, which has FT

$$\begin{aligned} Z(f) &= -jY(f-f_c) + jY(f+f_c) \\ &= \begin{cases} -X(f-2f_c), & 2f_c < f < 2f_c + f_M \\ +X(f), & -f_M < f < 0 \\ +X(f), & 0 < f < f_M \\ -X(f+2f_c), & -2f_c - f_M < f < -2f_c \\ 0, & \text{else} \end{cases} \end{aligned}$$

If $z(t)$ is then passed through a lowpass filter, with cutoff at $f = \pm f_M$, then the resulting signal is identical to $x(t)$.

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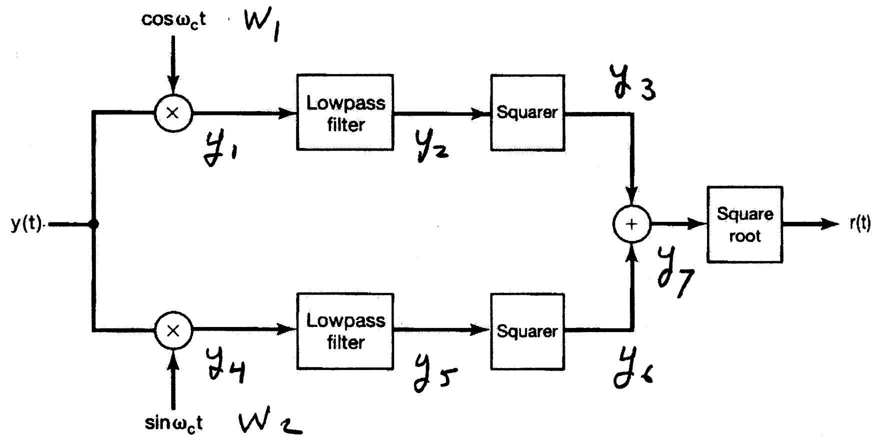
Problem S17 Solution

To begin, label the signals as shown below:



From the problem statement,

$$y(t) = [x(t) + A]\cos(2\pi f_c t + \theta_c)$$

Define

$$\begin{aligned} z(t) &= x(t) + A \\ w(t) &= \cos(2\pi f_c t + \theta_c) \end{aligned}$$

The factor $w(t)$ can be expanded as

$$w(t) = \cos(2\pi f_c t + \theta_c) = \cos\theta_c \, \cos 2\pi f_c t - \sin\theta_c \, \sin 2\pi f_c t$$

The Fourier transform of $w(t)$ is then given by

$$\begin{aligned} W(f) &= \mathcal{F}[\cos(2\pi f_c t + \theta_c)] \\ &= \frac{1}{2}\cos\theta_c \, [\delta(f - f_c) + \delta(f + f_c)] - \frac{1}{2}\sin\theta_c \, [-j\delta(f - f_c) + j\delta(f + f_c)] \\ &= \frac{1}{2}(\cos\theta_c + j\sin\theta_c)\,\delta(f - f_c) + \frac{1}{2}(\cos\theta_c - j\sin\theta_c)\,\delta(f + f_c) \end{aligned}$$

The Fourier transform of $z(t) = x(t) + A$ is given by

$$Z(f) = \mathcal{F}[z(t)] = X(f) + A\delta(f)$$

$Z(f)$ is bandlimited, because $X(f)$ is, and of course the impulse function is bandlimited. So the FT of $y(t)$ is given by the convolution

$$\begin{aligned} Y(w) &= Z(f) * W(f) \\ &= \frac{1}{2}\left[(\cos\theta_c + j\sin\theta_c)\, Z(f - f_c) + (\cos\theta_c - j\sin\theta_c)\, Z(f + f_c)\right] \end{aligned}$$

Next, compute the spectra of $y_1(t)$ and $y_2(t)$. To do so, we need the spectra of $w_1(t)$ and $w_2(t)$:

$$\begin{aligned} W_1(f) = \mathcal{F}[w_1(t)] &= \mathcal{F}[\cos 2\pi f_c t] \\ &= \frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)] \\ W_2(f) = \mathcal{F}[w_2(t)] &= \mathcal{F}[\sin 2\pi f_c t] \\ &= \frac{1}{2}[-j\delta(f-f_c)+j\delta(f+f_c)] \end{aligned}$$

Then

$$\begin{aligned} Y_1(f) &= W_1(f) * Y(f) \\ &= \frac{1}{2}[Y(f-f_c)+Y(f-f_c)] \\ &= \frac{1}{4}[(\cos\theta_c + j\sin\theta_c)\, Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c)\, Z(f)] \\ &\quad + \frac{1}{4}[(\cos\theta_c + j\sin\theta_c)\, Z(f) + (\cos\theta_c - j\sin\theta_c)\, Z(f+2f_c)] \\ &= \frac{1}{2}\cos\theta_c\, Z(f) \\ &\quad + \frac{1}{4}[(\cos\theta_c + j\sin\theta_c)\, Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c)\, Z(f+2f_c)] \end{aligned}$$

Similarly,

$$\begin{aligned} Y_4(f) &= W_2(f) * Y(f) \\ &= \frac{1}{2}[-jY(f-f_c)+jY(f-f_c)] \\ &= \frac{-j}{4}[(\cos\theta_c + j\sin\theta_c)\, Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c)\, Z(f)] \\ &\quad + \frac{j}{4}[(\cos\theta_c + j\sin\theta_c)\, Z(f) + (\cos\theta_c - j\sin\theta_c)\, Z(f+2f_c)] \\ &= -\frac{1}{2}\sin\theta_c\, Z(f) \\ &\quad + \frac{1}{4}[(-j\cos\theta_c + \sin\theta_c)\, Z(f-2f_c) + (j\cos\theta_c + \sin\theta_c)\, Z(f+2f_c)] \end{aligned}$$

Now, when $y_1(t)$ and $y_4(t)$ are passed through the lowpass filters, the $Z(f-2f_c)$ and $Z(f+2f_c)$ terms are eliminated, and the $Z(f)$ terms are passed. Therefore,

$$\begin{aligned} Y_2(f) &= \frac{1}{2}\cos\theta_c\, Z(f) \\ Y_5(f) &= -\frac{1}{2}\sin\theta_c\, Z(f) \end{aligned}$$

and

$$
\begin{aligned}
y_2(t) &= \frac{1}{2} \cos\theta_c \, z(t) \\
y_5(t) &= -\frac{1}{2} \sin\theta_c \, z(t)
\end{aligned}
$$

After passing these signals through the squarers, we have

$$
\begin{aligned}
y_3(t) &= \frac{1}{4} \cos^2\theta_c \, z^2(t) \\
y_6(t) &= \frac{1}{4} \sin^2\theta_c \, z^2(t)
\end{aligned}
$$

$y_7(t)$ is the sum of these, so that

$$
\begin{aligned}
y_7(t) &= y_3(t) + y_7(t) \\
&= \frac{1}{4}\left[\cos^2\theta_c \, z^2(t) + \sin^2\theta_c \, z^2(t)\right] \\
&= \frac{1}{4} z^2(t)
\end{aligned}
$$

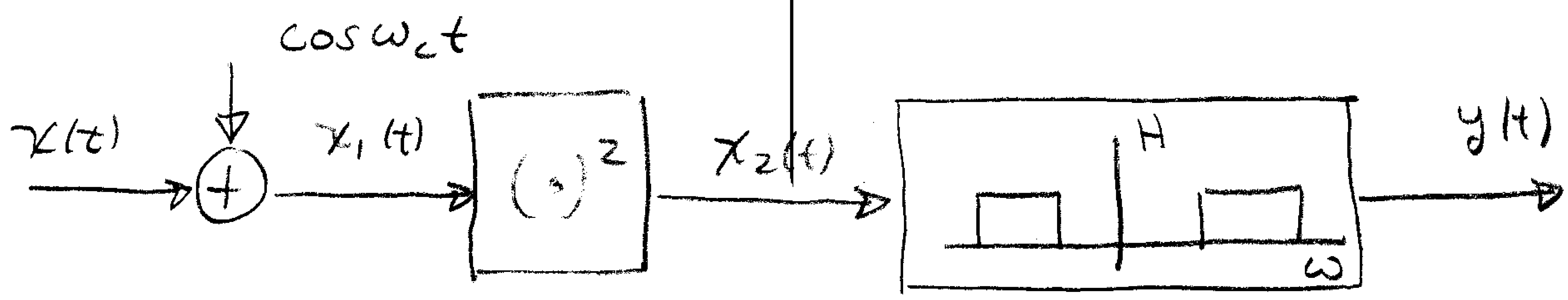
Finally, $r(t)$ is obtained by passing taking the square root of $y_7(t)$, so that

$$
\begin{aligned}
r(t) &= \sqrt{z^2(t)/4} \\
&= \frac{|z(t)|}{2}
\end{aligned}
$$

if the positive root is always taken. But $z(t) = x(t) + A$ is always positive, according to the problem statement. Therefore,

$$
x(t) = 2r(t) - A
$$

Redraw the block diagram:



Take each signal in turn:

$$x_1(t) = x(t) + \cos \omega_c t$$

$$\Rightarrow X_1(f) = X(f) + \frac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right)$$

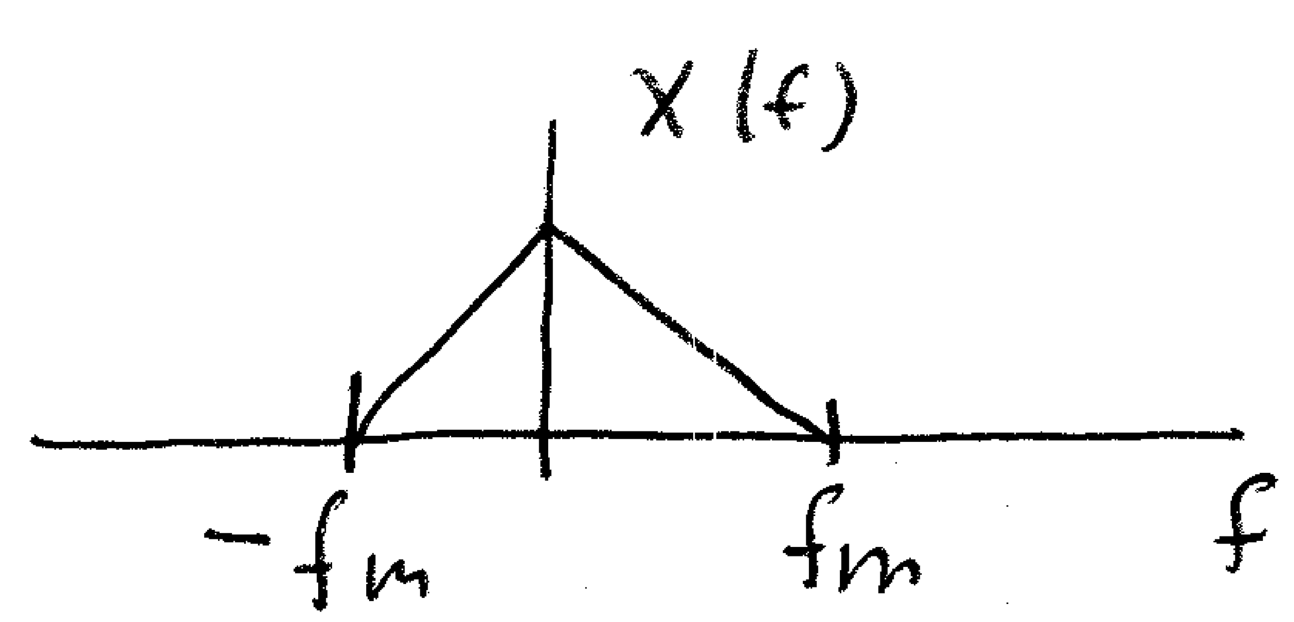
where $f_c = \omega_c / 2\pi$

$x_2(t)$ is $x_1^2(t)$, so

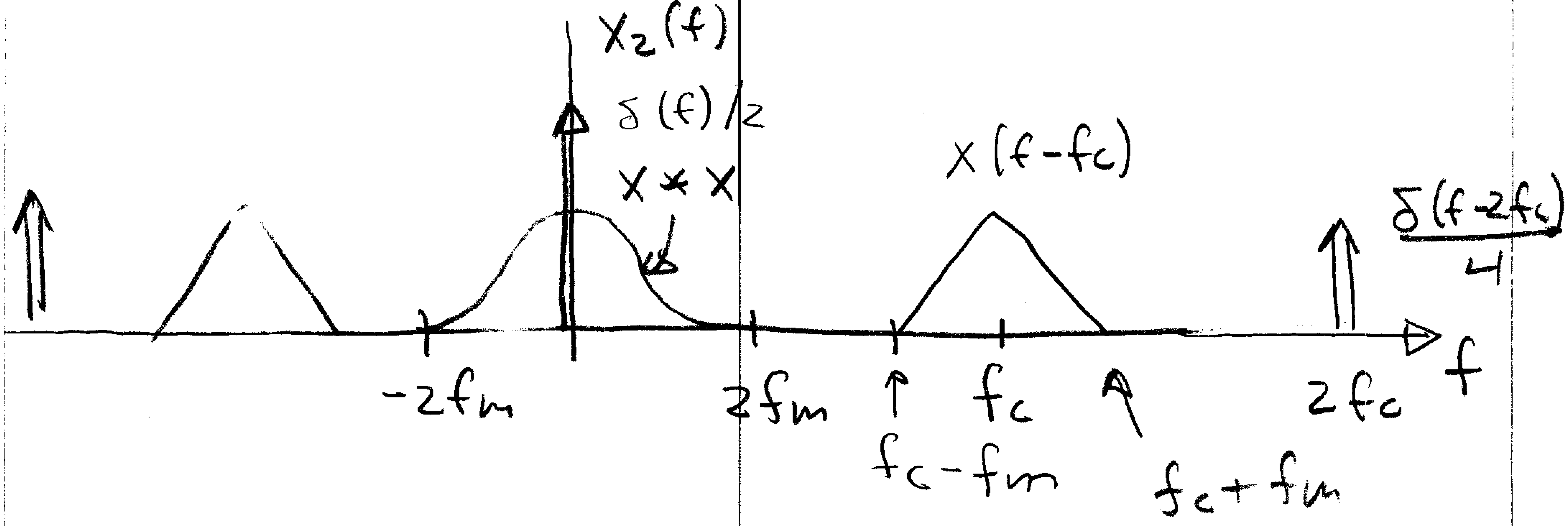
$$X_2(f) = X_1(f) * X_1(f)$$

$$= X(f) * X(f) + X(f-f_c) + X(f+f_c)$$
$$+ \frac{1}{4}\left[\delta(f-2f_c) + \delta(f+2f_c)\right]$$
$$+ \frac{1}{2}\delta(f)$$

Suppose $X(f)$ is



What does $X_2(f)$ look like?



Therefore, if we want

$$y(t) = x(t)\cos\omega_c t$$

$$\Rightarrow Y(f) = \frac{X(f-f_c)}{2} + \frac{X(f+f_c)}{2}$$

then we can take

$$f_\ell = f_c - f_m \qquad (\omega_\ell = \omega_c - \omega_m)$$

$$f_h = f_c + f_m \qquad (\omega_h = \omega_c + \omega_m)$$

$$A = 1/2$$

We also require that

$$f_c - f_m > 2f_m$$

$$\Rightarrow f_c > 3f_m$$

in order to have no overlap