

## 18.03SC Differential Equations, Fall 2011

### Transcript – Manipulating Fourier Series

PROFESSOR: Hi everyone. Welcome back.

So today I'd like to look at a problem on a manipulating Fourier series. Specifically, we're asked to find the Fourier series of  $\cos(2t - \pi/4)$ . And then the second problem is given a square wave function, which takes on the value of minus 1 and 1, and it's  $2\pi$  periodic. We're also told that the square wave function has the Fourier series  $\frac{4}{\pi} \sum \frac{1}{n} \sin nt$ . And the question is to find the Fourier series of three related functions.

So the first function takes on the value of 0 and 4 on the intervals  $-\pi$  to 0 and 0 to  $\pi$ . Note that this function is also  $2\pi$  periodic. The second problem it is to find the Fourier series of a function which is minus 1 from minus 1 to 0, and 1 from 0 to 1. So this is the square wave function, but now has period two.

And then lastly we're asked to find the Fourier series of a function  $f$  of  $t$ , which is absolute  $t$ . And it's defined as absolute  $t$  on the interval from minus  $\pi$  to  $\pi$ . However this function  $f$  is going to be extended on the entire domain with period  $2\pi$ . So I'll let you think about this problem and I'll be back in a moment.

Hi everyone. Welcome back. So let's take a look at the first problem, finding the Fourier series of  $\cos(2t - \pi/4)$ . So in this class when we say find the Fourier series, what we're really looking for are coefficients  $a_0$ ,  $a_n$ , and  $b_n$ , such that, for example, we can write our function out as a sum of sine and cosines with these values  $a_n$  and  $b_n$ .

Now, what we could do is we could take this function, plug it into the integral formulas, and compute using integrals, the value  $a_0$ ,  $a_n$ , and  $b_n$ . However, there's a bit of a trick. And that's to realize that this function,  $\cos(2t - \pi/4)$  is really just a sinusoidal function with a single frequency. And it's currently in an amplitude phase form.

And we can always convert an amplitude phase form of a sinusoidal function into some amplitude times a sine and some amplitude times a cosine and add the two up. So specifically, this can be done by recalling the cosine identity,  $\cos(A - B)$ , is  $\cos A \cos B + \sin A \sin B$ .

And if we identify  $A$  with  $2t$  and  $B$  with  $\pi/4$ , we obtain that  $f$  of  $t$  is  $\cos 2t \cos \pi/4 + \sin 2t \sin \pi/4$ . And now  $\sin \pi/4$  and  $\cos \pi/4$  are both  $1/\sqrt{2}$ . So  $f$  of  $t$  becomes  $1/\sqrt{2} \cos 2t + 1/\sqrt{2} \sin 2t$ . And that's it.

So when we take a look at this expression for  $f$  of  $t$ , we see that this is actually in exactly the same form that we were seeking in the beginning, the Fourier series representation for  $f$ . So this is the end of the problem. We don't have to actually compute any integrals.

So now let's take a look at part two. So for part two, we're asked to find the Fourier series of several different functions. And I'll just take a look at part one.

So in this case,  $f$  of  $t$  takes on the value of 0 from minus  $\pi$  to zero, and the value of 4 from 0 to  $\pi$ . And if I were to quickly sketch this function, it would look like a square wave, 0,  $\pi$ , minus  $\pi$ . This is  $f$  of  $t$ .

And we see that it's actually related-- I can draw in dashes just to be clear-- what the original square wave function might look like. So this is square wave of  $t$ . We see that if we were to take the square wave function and shift it up, the square wave takes on values of plus 1 and minus 1. But if we take the square wave function and shift it up 1 unit, and then rescale it, we'll then get the function we're looking for,  $f$  of  $t$ .

So specifically, we can write  $f$  of  $t$  in terms of the square wave as well. We have to shift the square wave function up 1, and then we have to amplify it by a factor of 2. So then when we seek the Fourier series for  $f$ , all we have to do is substitute the Fourier series for the square wave function  $n$ .

So let's do this. So we have  $1 + 4/\pi$ . So I'm writing here this Fourier series for the square wave function,  $n$  odd  $1/n$  sine  $nt$ . So by manipulating the Fourier series for the square wave function, we can arrive at a Fourier series for  $f$  of  $t$ ,  $n$  odd  $1/n$  sine  $nt$ . So this is the first function.

The second function  $f$  of  $t$ -- I'll just rewrite it again--  $f$  of  $t$  takes on the value of negative 1 from negative 1 to 0, and takes on the value of plus 1 from 0 to 1. So it's somewhat useful to draw, just to quickly sketch this function. And I'll draw in my axes here. So this is  $f$  of  $t$ . This is  $t$ .

And it transitions. It makes its jumps at the integers. So for example, it jumps from plus 1 to minus 1 at 1, and from plus 1 to minus 1 at minus 1 as well. And if I were to just quickly sketch the square wave function underneath of it, the square wave function makes its transitions at minus  $\pi$  and  $\pi$ . So this is the square wave function. This is  $t$ .

So we see that we can take the square wave function and squeeze it to obtain this function  $f$  of  $t$ . And the question now is, what factor do we have to squeeze the square wave function by to get  $f$  of  $t$ . Well, usually what I like to do is take one characteristic point in the sketch for in this case the square wave function. So a nice characteristic point is when the square wave function has a 0, so when it goes through the origin, and compare it to the same point on this function  $f$  of  $t$ . So we see that the point  $\pi$  has to get stretched to the point 1. So we basically need to take the time axis in the square wave function and squeeze it by a factor of  $\pi$ .

So what this means is that  $f$  of  $t$  is going to be the square wave. And when we do what mathematicians sometimes called a dilatation, a stretch or a squeeze. If we're squeezing by some factor, that factor always appears in the argument of the function we're squeezing. If it's a stretch, then it would be  $t$  over the stretch factor. And we can quickly check here that if  $t$  equals 1, so as  $t$  goes from 0 to 1, so when  $t$  is 1, the square wave function is being evaluated at  $\pi$ .

And I'll now write out the Fourier series. So this is going to be  $4/\pi$  sum of  $n$  odd  $1/n$  sine of  $n\pi t$ . And there's another quick check we can do. And that's this function  $f$  we were told is periodic with period 2. And if we just take a look at this function here, the Fourier series representation of it, we see that each

$\sin n\pi t$  is in fact periodic with period 2. So that's just another quick, back of the envelope check that we've done our calculation correctly.

So this concludes part two. And then lastly then for part three,  $f$  of  $t$  is the absolute value of  $t$  from  $-\pi$  to  $\pi$ . And it's extended beyond  $-\pi$  to  $\pi$  to make it  $2\pi$  periodic. And at first it might look like this function  $f$  of  $t$  is not related to the square wave function.

But if we look at it a little more carefully, and analyze the derivative of  $f$  of  $t$ , we see that this function takes on the value plus 1 on the interval of 0 to  $\pi$  and takes on the value of minus 1 on the interval of  $-\pi$  to 0. And this is exactly the square wave. So what we can do is we can write  $f$  of  $t$ . It's going to be an integral of the square wave function.

So specifically, we have the integral of square wave, which is going to be the integral of  $\frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nt$ . And I'm going to leave this as an indefinite integral. So when we integrate, we have a constant of integration  $c$  plus  $\frac{4}{\pi} \sum \frac{1}{n^2} \cos nt$ . And we have negative cosine  $nt$ . And this is  $n$  odd again.

And now we have one last question to answer. What is the constant of integration? Well, notice how the constant of integration can be identified with the  $a_0$  term of a Fourier series. So if we want the constant of integration, we can just calculate it directly by the integral formula for the  $a_0$  of this function. So what this means is  $c$  is going to be  $\frac{1}{2\pi}$ . Essentially it's just the average of the function, but it's  $-\pi$  to  $\pi$ , the absolute value of  $t$ , which is  $\frac{1}{\pi} \int_{-\pi}^{\pi} t dt = \frac{\pi^2}{2}$ . So we get  $\pi/2$ .

So just to quickly recap, in this problem we were given several functions which were related to the square wave function. And we started off knowing what the Fourier series for the square wave function was. And by manipulating or rewriting these functions in terms of the square wave function, we were able to compute their Fourier series. And we were able to do this essentially without having to evaluate any integrals.

OK. So I hope you enjoyed this problem, and I'll see you next time.

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